McGill University
Solution Sheet for MATH 247 Assignment 1

1. (a) We have to verify the vector space axioms for the operations $\oplus, \odot$:
(I) **Associative Law:** $((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3) = (x_1 + x_2 - 1, y_1 + y_2 + 2) \oplus (x_3, y_3) = (x_1 + x_2 + x_3 - 2, y_1 + y_2 + y_3 + 4) = (x_1, y_1) \oplus (x_2 + x_3 - 1, y_2 + y_3 + 2) = (x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3))$.
(II) **Commutative Law:** $(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2 - 1, y_1 + y_2 + 2) = (x_2 + x_1 - 1, y_2 + y_1 + 2) = (x_2, y_2) \oplus (x_1, y_1)$.
(III) **Existence of Zero:** The zero element is $(1, -2)$ since $(x, y) \oplus (1, -2) = (x + 1 - 1, y - 2 + 2) = (x, y)$.
(IV) **Existence of Additive Inverses:** The additive inverse of (x, y) is $(-x + 2, -y - 4)$ since $(x, y) \oplus (-x + 2, -y - 4) = (x - x + 2 - 1, y - y - 4 + 2) = (1, -2)$.
(V) **Multiplication by 1:** $1 \odot (x, y) = (x + 1 - 1, y + 2 - 2) = (x, y)$.
(VI) **Associativity of Scalar Multiplication:** $a \odot (b \odot (x, y)) = a \odot (bx + 1 - b, by + 2b - 2) = (abx + a - ab + 1 - a, aby + 2ab - 2a + 2a - 2) = (abx + 1 - ab, aby + 2ab - 2) = (ab) \odot (x, y)$.
(VII) **Distributive Laws:** (a) $(a + b)(x, y) = ((ax + bx + 1 - a - b, ay + by + 2a + 2b - 2) = (ax + 1 - a, ay + 2a - 2) \oplus (by + 1 - b, by + 2b - 2) = a \odot (x, y) \oplus b \odot (x, y)$;
(b) $a \odot ((x_1, y_1) \oplus (x_2, y_2)) = a \odot (x_1 + x_2 - 1, y_1 + y_1 + 2) = (ax_1 + ax_2 - a + 1 - a, ay_1 + ay_1 + 2a + 2a - 2) = (ax_1 + 1 - a, ay_1 + 2a - 2) \oplus (ax_2 + 1 - a, ay_2 + 2a - 2) = a \odot (x_1, y_1) \oplus a \odot (x_2, y_2)$.

(b) To show that T is an isomorphism of $(\mathbb{R}^2, \oplus, \odot)$ to $(\mathbb{R}^2, +, \cdot)$ we show
(i) T is **injective:** $T(x_1, y_1) = T(x_2, y_2) \implies (x_1 - 1, y_1 + 2) = (x_2 - 1, y_2 + 2) \implies x_1 - 1 = x_2 - 1$, $y_1 + 2 = y_2 + 2 \implies x_1 = x_2, y_2 = y_2 \implies (x_1, y_1) = (x_2, y_2)$.
(ii) T is **surjective:** $(x, y) = T(x - 1, y + 2)$
(iii) T is **linear:** $T(a \odot (x_1, y_1) \oplus b \odot (x_2, y_2) = T(ax_1 + bx_2 + 1 - a - b, ay_1 + by_2 + 2a + 2b - 2) = (ax_1 + bx_2 - a - b, ay_1 + by_2 + 2a + 2b) = (ax_1 - a, ay_1 + 2a) + (bx_2 - b, by_2 + 2b) = a(x_1 - 1, y_1 + 2) + b(x_2 - 1, y_2 + 2) = aT(x_1, y_1) + bT(x_2, y_2)$.

2. (a) $0 \in W$ since $x_n = 0$ for all $n \geq 1$ implies $x_{n+3} = x_{n+2} + nx_n = 0$ for all $n \geq 1$. If $a, b \in \mathbb{F}$, $x, y \in W$ then $(ax + by)_{n+3} = ax_{n+3} + by_{n+3} = ax_{n+2} + nbx_n + ay_{n+2} + nby_n = (ax + by)_{n+2} + n(ax + by)_n$ which implies $ax + by \in W$. Hence W is a subspace of V.

(b) $0 \in W$ since $f(x) = 0$ for all x implies $f(x) = f(2x^2 + 1) = 0$ for all x. If $a, b \in \mathbb{F}, f, g \in W$ then $(af + bg)(x) = af(x) + bg(x) = af(2x^2 + 1) + bg(2x^2 + 1) = (af + bg)(2x^2 + 1)$ which implies $af + bg \in W$. Hence W is a subspace of V.

(c) $0 \in W$ since $f(x) = 0$ for all x implies $f'(x) = f''(x) = 0$ for all x and hence that $f''(x) + xf'(x) + f(x) = 0$ for all x. If $a, b \in \mathbb{F}, f, f \in W$ then $(af + bg)''(x) + x(af + bg)'(x) + (af + bg)(x) = af''(x) + bg''(x) + x(af'(x) + bg'(x)) + af(x) + bg(x) = a(f''(x) + xf'(x) + f(x)) + b(g''(x) + xg''(x) + g(x)) = 0$ so that W is a subspace of V. Note that if $f \in \mathbb{C}^{\mathbb{R}}$ then $f = f_1 + if_2$ with $f_1, f_2 \in \mathbb{R}^{\mathbb{R}}$ and $f' = f_1' + if_2'$. If $a = a_1 + ia_2, b = b_1 + ib_2$ with $a_i, b_i \in \mathbb{R}$ and $g = g_1 + ig_2$ with $g_1, g_2 \in \mathbb{R}^{\mathbb{R}}$ we have $af + bg = (a_1 f_1 - a_2 f_2 + b_1 g_1 - b_2 g_2) + i(a_1 f_2 + a_2 f_1 + b_1 g_2 + b_2 g_1)$ which implies that $(af + bg)' = af' + bg'$.

(d) If $x, y \in \mathbb{R}$ with $x^2 - xy + y^2 = 0$ then $x = y = 0$ (complete the square). Hence $W = \{(a, b, 0) \mid a, b \in \mathbb{R}\}$ which is a subspace of V if $\mathbb{F} = \mathbb{R}$. If $\mathbb{F} = \mathbb{C}$ then W is not a subspace since $(1, 0, 0) \in W$ but $i(1, 0, 0) = (i, 0, 0) \notin W$ since $i \notin \mathbb{R}$.

(e) The zero matrix 0 is in W since $0^t = \overline{0} = 0$. If $X, Y \in W$ then $(X + Y)^t = X^t + Y^t = \overline{X} + \overline{Y} = \overline{X + Y}$ which implies that $X + Y \in W$. If $\mathbb{F} = \mathbb{R}$ and $a \in \mathbb{R}, X \in W$ then $\overline{aX} = a\overline{X} = aX$ which implies that W is a subspace of V if $\mathbb{F} = \mathbb{R}$. If $\mathbb{F} = \mathbb{C}$ we have $X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is in W but $iX = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$ is not so that W is not a subspace of V in this case.

3. (a) We have $f \in V_{\text{even}} \iff f(-x) = f(x)$ for all x and $f \in V_{\text{odd}} \iff f(-x) = -f(x)$ for all x. For any $f \in V$ we have $f = g + h$ where $g(x) = (f(x) + f(-x))/2$, $h(x) = (f(x) - f(-x))/2$. Since g is even and h is odd we have $V = V_{\text{even}} + V_{\text{odd}}$. The sum is direct since $V_{\text{even}} \cap V_{\text{odd}} = \{0\}$ since $f(x) = f(-x)$, $f(-x) = -f(x)$ implies $-f(x) = f(x)$ which implies $f(x) = 0$.

(b) We have $\exp(x) = e^x = (e^x + e^{-x})/2 + (e^x - e^{-x})/2 = \cosh(x) + \sinh(x)$so that $\exp = \cosh + \sinh$ is the decomposition of the exponential function exp as the sum of an even function and an odd function.

4. (a) Let $W = \{v \in V \mid v \in W_i \text{ for all } x \in I\}$ be the intersection of the family of subspaces W_i $(i \in I)$. The Then $0_V \in W$ since $0_V \in W_i$ for all i. If a, b are scalars and $u, v \in W$ then $u, v \in W_i$ for all i which implies that $au + bv \in W_i$ for all i and hence that $au + bv \in W$.

(b) The statement is false since $\mathbb{R}^2 = \mathbb{R}(1, 0) \oplus \mathbb{R}(0, 1) = \mathbb{R}(1, 0) \oplus \mathbb{R}(1, 1)$ but $\mathbb{R}(0, 1) \neq \mathbb{R}(1, 1)$.