McGill University

MATH 247 Midterm Test

**Justify All Your Assertions**

**Determinants not allowed**

1. Determine which of the following subsets $S$ of the given real vector space $V$ are subspaces of $V$.

   (a) $V = \mathbb{R}^\infty$, $S = \{x = (x_1, x_2, \ldots, x_n, \ldots) \in V \mid x_{n+2} = x_n x_{n+1} \text{ for } n \geq 1\}$.

   (b) $V = \mathbb{R}^{2\times 2}$, $S = \{X \in V \mid X \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} X\}$.

2. Determine whether the following sequences are linearly independent or linearly dependent.

   (a) $(e^x, e^{2x}, e^{3x}, e^{4x})$ in $\mathbb{R}^\mathbb{R}$.

   (b) $(A, A^2, A^3, A^4, A^5)$ where $A = \begin{bmatrix} 11 & 12 \\ 13 & 14 \end{bmatrix} \in \mathbb{R}^{2\times 2}$.

3. Let $V$ be the real vector space of polynomial functions $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ and let $T : V \to V$ be the mapping defined by

$$T(f)(x) = f(x) + f(-x).$$

   (a) Prove that $T$ is linear.

   (b) Find bases for the kernel and image of $T$.

4. Let $A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

   (a) Find a basis for $\mathbb{R}^{4\times 1}$ consisting of eigenvectors of $A$. Hint: $(A - I)^2 = 4(A - I)$.

   (b) Find an invertible matrix $P \in \mathbb{R}^{4\times 4}$ such that $P^{-1}AP$ is a diagonal matrix.