McGill University
Math 247: Honours Applied Linear Algebra
Assignment 6: due Monday, April 10, 2006

1. (a) Find the spectral decomposition of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

   **Hint:** By inspection 3 and 5 are eigenvalues of $A$. Explain why this is the case.

   (b) Using (a), solve the initial value problem

$$i\frac{dX}{dt} = Ax, \quad X(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

   (c) Using (a), find the maximum and minimum values of the quadratic form

$$q(X) = 3x_1^2 + 2x_1x_2 + 2x_1x_3 + 3x_2^2 + 2x_2x_4 + 3x_3^2 + 2x_3x_4 + 3x_4^2$$

   subject to the constraint $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ and find where these maxima and minima are attained.

2. Find the Jordan canonical form of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

   and find an invertible matrix such that $P^{-1}AP$ is in Jordan canonical form.

3. **(Bonus Question)** Find the spectral decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

4. **(Bonus Question)**

   (a) Prove or give a counterexample: the product of any two self-adjoint operators on an inner-product space is self-adjoint.

   (b) If $T$ is a normal operator on a finite-dimensional complex inner product space such that $T^9 = T^8$ prove that $T$ is self-adjoint and $T^2 = T$.

5. **(Bonus Question)**

   (a) Let $T$ be a linear operator on a vector space $V$ such that $\dim(\mathrm{Im}(T)) = n$. Prove that $T$ has at most $n + 1$ distinct eigenvalues.

   (b) If $T$ is a linear operator on a vector space $V$ such that any vector of $V$ is an eigenvector of $T$ prove that $T$ is a scalar multiple of the identity operator.