McGill University
Math 247: Honours Applied Linear Algebra
Assignment 5: due Wednesday, March 29, 2006

1. The trace of a square matrix $A = [a_{ij}] \in F^{n\times n}$ is the scalar

$$\mathrm{tr(A)} = \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{a_{i\,i}} = \mathrm{a_{11}} + \ldots + \mathrm{a_{nn}}.$$

   (a) Show that tr is linear and that $\mathrm{tr(AB)} = \mathrm{tr(BA)}$.

   (b) If $\phi : F^{n\times n} \to F$ is linear and $\phi(AB) = \phi(BA)$ for all $A, B \in F^{n\times n}$, show that $\phi = c\,\mathrm{tr}$ for some scalar c. **Hint:** Show that the subspace spanned by the matrices of the form $AB - BA$ has dimension $n^2 - 1$.

2. (a) Show that $< (x_1, x_2), (y_1, y_2) >= x_1y_1 + 2x_1y_2 + 2x_2y_1 + 5x_2y_2$ defines an inner product on $\mathbb{R}^2$.

   (b) Show that $< (x_1, x_2), (y_1, y_2) >= x_1\bar{y}_1 + ix_1\bar{y}_2 - ix_2\bar{y}_1 + 2x_2\bar{y}_2$ defines an inner product on $\mathbb{C}^2$.

3. Let $V = C_{\mathbb{R}}([0, 1])$ with inner product $< f, g >= \int_0^1 f(x)g(x)\,dx$.

   (a) Find the best approximation to $f(x) = \sin(\pi x)$ by a function in $\mathrm{Span}(1, x, x^2)$.

   (b) If ϕ is the linear form on $W = \mathrm{Span}(1, x, x^2)$ defined by $\phi(f) = f(1)$, find $g \in W$ such that $\phi(f) =< f, g >$ for all $f \in W$.

4. Let $V = \mathbb{C}^{n\times n}$ with the inner product $< X, Y >= \mathrm{tr}(\mathrm{X}\overline{\mathrm{Y}}^{\mathrm{t}})$ and let T be the linear operator on V defined by

$$T(X) = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} X - X \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}.$$

   (a) Show that T is self-adjoint.

   (b) Find an orthonormal basis of V consisting of eigenvectors of T. **Hint:** Use the fact that $\mathrm{Im(T)}^{\perp} = \mathrm{Ker(T)}$.

   (c) Find the matrix A of T with respect to the standard basis of V and find a unitary matrix U such that $U^{-1}AU$ is a diagonal matrix.