McGill University
Math 247: Honours Applied Linear Algebra
Assignment 4: due Friday, March 16, 2006

1. **(Bonus Question)** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

   (a) Show that $A^2 - (a+d)A + (ad-bc)I = 0$.

   (b) Show that $x^2 - (a+d)x + (ad-bc)$ is the minimal polynomial of A if $A \neq \lambda I$ for any scalar λ.

   (c) Show that the minimal polynomial of A is $(x-\lambda)^2$ if and only if A is similar to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

2. Two companies A and B compete for market share of their product. Suppose that that the first month they both have the same market share but that after the second month half of the purchasers of A's product switched to B's product and 1/3 of the purchasers of B's product switch ed to A's product. If x_n, y_n are the market shares for A's and B's product respectively after n months and the above trend continues, show that

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}.$$

   Find the limiting market shares, i.e., find $\lim x_n, \lim y_n$ as $n \to \infty$. **Hint:** Look at the following bonus question.

3. **(Bonus Question)** The real matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is said to be a stochastic matrix if $a, b, c, d \geq 0$ and $a + c = b + d = 1$. If $a, b, c, d > 0$ show that

$$\lim_{n\to\infty} A^n = \begin{bmatrix} \alpha & \alpha \\ \beta & \beta \end{bmatrix}$$

   where $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is the unique eigenvector of A with eigenvalue 1 with $\alpha + \beta = 1$. Hint: Show that $A = PDP^{-1}$ with D diagonal and use the fact that $A^n = PD^nP^{-1}$.

4. (a) If T is a linear operator on a vector space V with $T^n = 0$ $(n \geq 1)$, show that $1 - T$ is invertible with

$$(1-T)^{-1} = 1 + T + T^2 + \cdots + T^{n-1}.$$

   (b) Use (a) to show that if $(T-a)^n = 0$ and $b \neq a$ then $T - b$ is invertible with

$$(T-b)^{-1} = (a-b)^{-1}(1 + \frac{1}{b-a}(T-a) + \frac{1}{(b-a)^2}(T-a)^2 + \cdots + \frac{1}{(b-a)^{n-1}}(T-a)^{n-1}).$$

   (c) Use (b) to solve $(D-1)^2 y = xe^{2x}$ on $\mathrm{Ker(D-2)^2}$.

   (d) Use (c) to find all solutions of $(D-1)^2 y = xe^{2x}$. **Hint:** If y_P is a solution of $(D-1)^2 y = xe^{2x}$ show that $y - y_P \in \mathrm{Ker((D-1)^2)}$.

5. Let $A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 \\ 0 & -2 & -1 & -2 \\ -2 & 0 & -2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 & 2 & 2 \\ 3 & 4 & 2 & 2 \\ -1 & 1 & 1 & 2 \\ -1 & -3 & 0 & -1 \end{bmatrix}$.

   Show that $AB = BA$ and use this to find a single matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices, i.e., A and B are simultaneously diagonalizable. **Hint:** Look at the following bonus question and notice that A is the matrix in question 4 of assignment 3.

6. **(Bonus Question)** Let S, T be linear operators on a finite dimensional vector space V and suppose that $ST = TS$.

   (a) Show that $\mathrm{Ker(p(S))}$ and $\mathrm{Im(p(S))}$ are T-invariant for any polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$.

   (b) If T is diagonalizable and W is a T-invariant subspace, show that the restriction of T to W is diagonalizable. **Hint:** Use the minimal polynomial criterion for diagonalizability.

   (c) If S is diagonalizable show that T is diagonalizable if and only if the restriction of T to each of the eigenspaces of S is diagonalizable in which case there is a basis of V consisting of vectors which are eigenvectors for both S and T.

7. (a) If L is the left shift operator on R^∞ and $a \neq 1$, show that

   $$\mathrm{Ker((L-a)^k) = Span((a^{n-1}), (na^{n-1}), \ldots, (n^{k-1}a^{n-1}))},$$

   where (x_n) denotes the sequence $(x_1, x_2, \ldots, x_n, \ldots)$.

   (b) Use (a) to find a formula for

   $$s_n = \sum_{k=1}^{n} k^2 3^k.$$

   Hint: If $s = (s_n)$, show that $s \in \mathrm{Ker((L-3)^3(L-1))}$.

8. Find the general solution of the system of differential equations

   $$\begin{aligned} \frac{dx_1}{dt} &= -2x_1 + x_2 \\ \frac{dx_2}{dt} &= x_1 - 2x_2 \end{aligned}$$

   by writing it in matrix form $\frac{dX}{dt} = AX$ and making the change of variable $X = PY$.