McGill University
Math 247: Honours Applied Linear Algebra
Assignment 3: due Wednesday, February 15, 2006

Midterm Test: Friday, March 3 (in class).

1. (a) If D is the differentiation operator on $V = C^\infty_{\mathbb{R}}(\mathbb{R})$ and $a \in \mathbb{R}$, prove that
$$\mathrm{Ker}((\mathrm{D}-\mathrm{a})^{\mathrm{n}}) = \mathrm{Span}(\mathrm{e}^{\mathrm{ax}}, \mathrm{xe}^{\mathrm{ax}}, \ldots, \mathrm{x}^{\mathrm{n}-1}\mathrm{e}^{\mathrm{ax}}).$$
   (b) If W is the solution space of the differential equation $f^{iv}(x) - 2f''(x) + f(x) = 0$, show that $W = \mathrm{Ker}((\mathrm{D}-1)^2) \oplus \mathrm{Ker}((\mathrm{D}+1)^2)$.
   (c) Find the solution of the differential equation in (b) satisfying
$$f(0) = 1, \quad f'(0) = 2, \quad f''(0) = 3, \quad f'''(0) = 4.$$

2. Let T be the linear operator on $V = \mathbb{R}^{2\times 2}$ defined by $T(A) = 2A + A^t$, where A^t is the transpose of A.
   (a) Find bases for the kernel and image of T.
   (b) Show that $T^2 - 4T + 3 = 0$ and use this to find the eigenvalues of T.
   (c) Find a basis of V consisting of eigenvectors of T.

3. (a) Find a 4×4 real matrix A such that
$$A\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad A\begin{bmatrix}1\\2\\1\\2\end{bmatrix} = \begin{bmatrix}1\\2\\1\\2\end{bmatrix}, \quad A\begin{bmatrix}1\\1\\2\\2\end{bmatrix} = 2\begin{bmatrix}1\\1\\2\\2\end{bmatrix}, \quad A\begin{bmatrix}1\\2\\3\\3\end{bmatrix} = 2\begin{bmatrix}1\\2\\3\\3\end{bmatrix}.$$
   Show that A is unique.
   (b) If A is the matrix in (a), find the eigenvalues of A and a basis for each eigenspace of A.

4. (a) Find a 4×4 real matrix A whose null space and column space are spanned by the column matrices
$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \quad \begin{bmatrix}4\\3\\2\\1\end{bmatrix}.$$
   (b) What are the eigenvalues of the matrix A found in (a)? Does $\mathbb{R}^{4\times 1}$ have a basis consisting of eigenvectors of A?

5. Let $A = \begin{bmatrix}1 & 2 & 2 & 2\\ 2 & 1 & 2 & 2\\ 0 & -2 & -1 & -2\\ -2 & 0 & -2 & -1\end{bmatrix}$.
   (a) Using the fact that $A^2 = I$, find the eigenvalues of A and a basis for each eigenspace of A.
   (b) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.