McGill University
Math 247: Honours Applied Linear Algebra
Assignment 2: due Friday, February 3, 2006

1. Let V be a vector space over a field F and let $v_1, v_2, v_3, v_4, w \in V$.

   (a) Prove that if (v_1, v_2, v_3, v_4) spans V then so does $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$.

   (b) Prove that if (v_1, v_2, v_3, v_4) is linearly independent then so is $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$.

   (c) Show that $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4 - v_1)$ is linearly dependent.

   (d) If (v_1, v_2, v_3, v_4) is linearly independent and $(v_1 + w, v_2 + w, v_3 + w, v_4 + w)$ is linearly dependent, show that $w \in \mathrm{Span}(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4)$

2. Determine whether the given sequence of real vectors are linearly independent or not. In the case of linear dependence find a non-trivial dependence relation.

   (a) $(1, 1, 1, 1), (1, 1, 2, 2), (2, 2, 1, 1), (1, 1, 3, 3), (4, 4, 3, 3) \in \mathbb{R}^4$.

   (b) $f(x) = \sin(x + 1), g(x) = \sin(x + 2), h(x) = \sin(x + 3) \in \mathbb{R}^{\mathbb{R}}$

   (c) $f(x) = e^x, g(x) = xe^{2x}, h(x) = x^2e^{3x} \in \mathbb{R}^{\mathbb{R}}$

   (d) $(1, 2, 3, \ldots, n, \ldots), (1, 2^2, 3^2, \ldots, n^2, \ldots), (1, 2^3, 3^3, \ldots, n^3, \ldots) \in \mathbb{R}^\infty$

3. (a) Find a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors in $2(a)$ and complete this basis to a basis of $\mathbb{R}^4$.

   (b) Show that the subspace in $3(a)$ is equal to $\mathrm{Span}((3, 3, 7, 7), (7, 7, 3, 3))$.

4. Let $U = \mathrm{Span}((2, 1, 1, 3), (1, 2, 2, 1), (3, 2, 1, 5))$, $V = \mathrm{Span}((1, 4, 3, 5), (1, 1, 2, 2), (4, 3, 1, 7))$. Find bases for $U + V$ and $U \cap V$.

5. Let U be the subspace of $\mathbb{R}^\infty$ defined by

$$U = \{x = (x_1, x_2, \ldots, x_n, \ldots) \in \mathbb{R}^\infty \mid x_{n+4} = 5x_{n+2} - 4x_n (n \geq 1)\}.$$

   Prove that the infinite sequences x, y, z, w defined by

$$x_n = 1, \quad y_n = (-1)^{n-1}, \quad z_n = 2^{n-1}, \quad w_n = (-2)^{n-1}$$

   form a basis for U. You may assume as known that $\dim(U) = 4$. Use this to find a formula for the n-th term of the infinite sequence $u \in U$ where $u_1 = 1, u_2 = 2, u_3 = 3, u_4 = 4$.