189-247A: Final Examination (1994)

1. Which of the following sequences of vectors $v_1, \ldots, v_3$ in the given vector space V over the given field K are linearly independent? In they are linearly dependent, give a dependence relation.

   (i) $V = \mathbb{R}^4$, $K = \mathbb{R}$, $v_1 = (1, -1, 3, 1),\ v_2 = (-2, 1, -2, -3),\ v_3 = (1, 0, -1, 2)$;

   (ii) $V = \mathbb{C}^3$, $K = \mathbb{C}$, $v_1 = (1+i, 2-i, i),\ v_2 = (2-i, i, 1+i),\ v_3 = (2+3i, 2+i, 1-i)$;

   (iii) $V = C^\infty(\mathbb{R})$, $K = \mathbb{R}$, $v_1 = \sin(x),\ v_2 = x\cos(x),\ \sin(2x)$.

2. (a) Which of the following subsets of $\mathbb{R}^\infty$ are subspaces?

   (i) $S = \{x = (x_1, x_2, ..., x_n, ...) \mid x_{n+2} = x_{n+1} + nx_n (n \geq 1)\}$;

   (ii) $S = \{x = (x_1, x_2, ..., x_n, ...) \mid x_{n+1} = (n-1)x_n^2 (n \geq 1)\}$;

   (iii) $S = \{x = (x_1, x_2, ..., x_n, ...) \mid x_{n+1} = nx_n^2\}$.

   (b) Find a 4×4 matrix whose nullspace is spanned by the vectors

$$(1, 2, 2, 3)^T,\ (1, -1, 1, -1)^T.$$

3. (a) Explain carefully what is meant by the kernel and image of a linear mapping. Show that the kernel and image of a linear mapping $T : V \to W$ are respectively subspaces of V and W.
   (b) Show that the mapping $S : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined by $S(X) = X + X^T$ is linear and find bases for its kernel and image.

4. Let W be the solution space of the homogeneous system

$$x_1 + 2x_2 + 3_3 + 4_4 + x_5 = 0$$
$$2x_1 + 4x_2 + 3x_3 + 2x_4 + 3x_5 = 0.$$

   (a) Show that the vectors

$$(-3, 1, -1, 1, 1),\ (-4, 1, -3, 2, 3),\ (-1, 1, -1, -1, 1)$$

   are a basis of W.
   (b) Show that the mapping $T : W \to \mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3)$$

   is an isomorphism of vector spaces.

5. (a) Find an orthonormal basis for the subspace W of $\mathbb{R}^4$ spanned by

$$u_1 = [1, 0, 1, 1)]^T,\ u_2 = [0, 1, 0, 1]^T,\ u_3 = [1, 1, 0, 1]^T.$$

(b) Find the orthogonal projection w of $v = [1,1,1,1]^T$ on W. If $w' \in W$, what can you say about $||v - w'||_2$?
(c) Find a 4×3 matrix Q such that, for any $v \in \mathbb{R}^3$, the vector QQ^Tv is the orthogonal projection of v on W.

6. Let $V = \mathbb{R}^{2\times 2}$ and let $T : V \to V$ be the linear mapping defined by

$$T(X) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} X - X \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

(a) Show that T is self-ajoint for the inner product on V defined by $< X, Y >= \mathrm{tr}(X^TY)$, where $\mathrm{tr}(A)$ is the trace of A.
(b) Find the matrix of T with respect to the usual basis of V. Why is this matrix necessarily symmetric?
(c) Find the rank and nullity of T.

7. Let $A = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 3 & 2 \\ -2 & 2 & 3 \end{bmatrix}$.
(a) Find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix.
(b) Find symmetric 3×3 matrices P_1, P_2 such that $P_1^2 = P_1$, $P_2^2 = P_2$, $P_1P_2 = P_2P_1 = 0$ and $A = -P_1 + 5P_2$.
(c) Using (b), compute A^{-1} and e^{tA} for any real number t.