189-247B: Final Examination (1991)

1. (a) If W is a subspace of an inner product space V define what is meant by the orthogonal complement $W^\perp$ of W in V. Prove that $W^\perp$ is a subspace of V.

   (b) If $A \in \mathbb{R}^{n\times n}$, show that the orthogonal complement of the (right) nullspace of A is the column space of A^t (A^t = transpose of A).

2. If W is a subspace of the vector space V, then vectors $v_1, \ldots, v_n \in V$ are said to be linearly independent mod W if $a_1v_1 + \cdots + a_nv_n \in W$ implies $a_1 = \cdots = a_n = 0$.

   (a) If T is a linear operator on a vector space V such that $v_1, ..., v_n \in \mathrm{Ker}(T^2)$ are linearly independent mod $\mathrm{Ker}(T)$, show that $T(v_1), ..., T(v_n)$ are linearly independent.

   (b) If $A = \begin{bmatrix} 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, show that $A^2 = 0$ and find an invertible 5×5 matrix P with $P^{-1}AP$ in Jordan canonical form.

3. Let T be the linear operator on $\mathbb{R}^{2\times 2}$ defined by $T(A) = A - A^t$.

   (a) Find the matrix of T with respect to the usual basis of $\mathbb{R}^{2\times 2}$.

   (b) Find the characteristic and minimal polynomials of T. Is T diagonalizable?

4. Let $V = \mathrm{Ker}(D^3 - D^2)$, where D is the differentiation operator on $C^\infty_{\mathbb{R}}(\mathbb{R})$. Show that the mapping $T : V \to \mathbb{R}^3$ defined by $T(f) = (f(0), Df(0), D^2f(0))$, is an isomorphism of vector spaces.

5. If A is the real symmetric matrix $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$, find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix. Use this to find symmetric matrices P_1, P_2, P_3, P_4 with $P_i^2 = P_i$ for $1 \le i \le 4$, $P_iP_j = 0$ for $i \ne j$ and

$$A = P_1 + P_2 - P_3 + 3P_4.$$