189-247B Final Examination (1990)

1. (a) In each of the following decide whether or not W is a subspace of the vector space V over the real field $\mathbb{R}$.

   (i) $V = \mathbb{R}^\infty$, $W = \{ (x_1, x_2, \ldots, x_n, \ldots) \in V \ : \ x_{n+4} = 4x_{n+3} - x_{n+1} + 8x_n \ \text{ for } \ n \geq 1\}$;

   (ii) $V = \mathbb{R}^\infty$, $W = \{ (x_1, x_2, \ldots, x_n, \ldots) \in V \ : \ x_{n+2} = x_n x_{n+1} \ \text{ for } \ n \geq 1\}$;

   (iii) $V = C^\infty(\mathbb{R})$, $W = \{ f \in V \ : \ f^{iv}(x) + f'(x) + f(x) = 0 \ \text{ for all } \ x \in \mathbb{R}\}$.

   (b) In each case that W is a subspace of V find a linear operator on V whose kernel is W.

2 Find a linear operator on $\mathbb{R}^4$ whose kernel and image are generated by the vectors

$$(1, 1, 1, 1), \ (1, -1, 1, -1).$$

What is the matrix of this operator with respect to the usual basis of $\mathbb{R}^4$.

3. If $T : \text{Ker}(D - aI)^n \to \mathbb{R}^n$ is the mapping defined by

$$T(f) = (f(0), f'(0), ..., f^{(n-1)}(0))$$

show that T is an isomorphism of vector spaces.

4. Let $V = \mathbb{R}^{2\times 2}$ and let $T : V \to V$ be the linear operator on V defined by

$$T(X) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} X - X \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

   (a) Show that $T^3 = 36T$;

   (b) Find the matrix of T with respect to the usual basis of V;

   (c) Find the characteristic and minimum polynomials of T;

   (d) Is T diagonalizable? (Do not find a basis of eigenvectors.)

5. Find the Jordan canonical form of the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

over the field $\mathbb{F}_2$. Find the characteristic and minimum polynomials of T.

6. (a) If A is the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix.

   (b) Find the orthogonal projection of the vector $[1, 2, 3, 4]$ onto the left nullspace of the matrix A.