Math 247B: Linear Algebra
Solutions for Midterm Test

1. (a) The mapping $T : V \in V$ defined by $T(f)(x) = f(x^2)$ is linear since

$$T(af + bg)(x) = (af + bg)(x^2) = af(x^2) + bg(x^2) = (aT(f) + bT(g))(x).$$

Then $W_1 = \mathrm{Ker(I_V - T)}$ which is a subspace of V since $I_V - T$ is linear and the kernel of a linear mapping is linear.

(b) The constant function $f(x) = 1$ ($x \in \mathbb{R}$) is in W_2 but $g = 2f \notin W_2$ since $g(x) = 2$ while $g(x)^2 = 4$.

2. (a) The sequence of vectors $v_1, v_2, \ldots , v_n$ is linearly independent if $a_1v + a_2v_2 + \cdots + a_nv_n = 0$ implies that the scalars $a_1, a_2, \ldots , a_n$ are all zero; they are linearly independent if there are scalars $a_1, a_2, \ldots , a_n$ not all zero with $a_1v + a_2v_2 + \cdots + a_nv_n = 0$.

(b) Since $D(e^{ax}) = ae^{ax}$ the given functions are eigenvectors of the differention operator with distinct eigenvalues and hence are linearly independent.

(c) The 5 given matrices are linearly independent since they lie in the 4-dimensional vector space $F^{2\times2}$, where F is the field of scalars. Theorem: Any m vectors in an n-dimesional vector space are linearly dependent if $m > n$.

3. (a) The mapping T is linear since

$$T(aX+bY) = A(aX+bY)-(aX+bY)A = a(AX-XA)+b(AY-YA) = aT(X)+bT(Y).$$

(b) If $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have $T(X) = \begin{bmatrix} 3(c+b) & 3(d-a) \\ 3(d-a) & -3(b+c) \end{bmatrix}$. Hence $T(X) = 0$ iff $b = -c$ and $a = d$ which implies that I, A is a basis for Ker(T). Since $\dim(\mathrm{Im(T)}) = 4 - \dim(\mathrm{Ker(T)}) = 2$ and $T(X) = 3(b+c)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 3(d-a)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ we see that $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is a basis for the image of T.

(c) If T is a linear operator on a finite dimensional vector space V we have $\dim(V) = \dim(\mathrm{Ker(T)}) + \dim(\mathrm{Im(T)})$.