McGill University
Math 247B: Linear Algebra
Solution Sheet for Assignment 6

1. The characteristic polynomial of A is λ^4. We have

$$\begin{bmatrix}1 & -1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\end{bmatrix}, \quad A^2 = \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 4 & 0 & 0 & 0\\ 0 & -4 & 0 & 0\end{bmatrix}, \quad A^3 = \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 4 & -4 & 0 & 0\\ -4 & 4 & 0 & 0\end{bmatrix}, \quad A^4 = 0$$

so that, if Null(A) denotes the null space of A, we have

$$\mathrm{Null(A)} = \mathbb{R}\mathrm{f}_1, \quad \mathrm{Null(A^2)} = \mathbb{R}\mathrm{f}_1 \oplus \mathbb{R}\mathrm{f}_2, \quad \mathrm{Null(A^3)} = \mathbb{R}\mathrm{f}_1 \oplus \mathbb{R}\mathrm{f}_2 \oplus \mathbb{R}\mathrm{f}_3, \quad \mathbb{R}^{4\times1} = \mathrm{Null(A^4)} = \mathbb{R}\mathrm{f}_1 \oplus \mathbb{R}\mathrm{f}_2 \oplus \mathbb{R}\mathrm{f}_3 \oplus \mathbb{R}\mathrm{f}_4$$

$$f_1 = \begin{bmatrix}0\\0\\1\\-1\end{bmatrix}, \quad f_2 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \quad f_3 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \quad f_4 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}.$$

Since $Af_1 = 0$, $Af_2 = f_1$, $Af_3 = 2f_1 + 3f_2$, $Af_4 = 2f_1 + 3f_2 + f_3$, the required matrix P is the matrix with columns f_1, f_2, f_3, f_4. We then have

$$P = \begin{bmatrix}0 & 0 & 1 & 1\\ 0 & 0 & -1 & 0\\ 1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\end{bmatrix}, \quad P^{-1}AP = \begin{bmatrix}0 & 1 & 2 & 2\\ 0 & 0 & 3 & 3\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Note that if P is the matrix with columns $A^3f_4, A^2f_4, Af_4, f_4$, we have

$$P = \begin{bmatrix}0 & 0 & 1 & 1\\ 0 & 0 & 1 & 0\\ 4 & 4 & 1 & 0\\ -4 & 0 & 1 & 0\end{bmatrix}, \quad P^{-1}AP = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

2. The characteristic polynomial of A is $\lambda^2(\lambda-2)^2$. We have

$$A = \begin{bmatrix}3 & -1 & 1 & 1\\ 1 & 1 & 1 & 2\\ 0 & 0 & 1 & 1\\ 0 & 0 & -1 & -1\end{bmatrix}, \quad A^2 = \begin{bmatrix}8 & -4 & 2 & 1\\ 4 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, \quad A^3 = \begin{bmatrix}20 & -12 & 5 & 1\\ 12 & -4 & 3 & 3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

$$\mathrm{Null(A)} = \mathbb{R}\mathrm{f}_1, \quad \mathrm{Null(A^2)} = \mathrm{Null(A^3)} = \mathbb{R}\mathrm{f}_1 \oplus \mathbb{R}\mathrm{f}_2, \quad \mathrm{f}_1 = \begin{bmatrix}-1/4\\-3/4\\-1\\1\end{bmatrix}, \quad \mathrm{f}_2 = \begin{bmatrix}1/4\\0\\-1\\0\end{bmatrix},$$

$$A - 2I = \begin{bmatrix}1 & -1 & 1 & 1\\ 1 & -1 & 1 & 2\\ 0 & 0 & -1 & 1\\ 0 & 0 & -1 & -3\end{bmatrix}, \quad (A-2I)^2 = \begin{bmatrix}0 & 0 & -2 & -3\\ 0 & 0 & -3 & -6\\ 0 & 0 & 0 & -4\\ 0 & 0 & 4 & 8\end{bmatrix}, \quad (A-2)^3 = \begin{bmatrix}20 & -12 & 5 & 1\\ 12 & -4 & 3 & 3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

$$\mathrm{Null(A-2I)} = \mathbb{R}\mathrm{f}_3, \quad \mathrm{Null((A-I)^2)} = \mathrm{Null((A-I)^3)} = \mathbb{R}\mathrm{f}_3 \oplus \mathbb{R}\mathrm{f}_4, \quad \mathrm{f}_3 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \quad \mathrm{f}_4 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}.$$

Since $Af_1 = 0$, $Af_2 = f_1$, $Af_3 = 2f_3$, $Af_4 = 2f_4 + f_3$, the required matrix P is the matrix whose columns are f_1, f_2, f_3, f_4. We then have

$$P = \begin{bmatrix} -1/4 & 1/4 & 1 & 1 \\ -3/4 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

3. An orthogonal basis for W is $f_1 = (1,0,0,1)$, $f_2 = (0,1,1,0)$ so that

$$P_W(v) = \frac{< v, f_1 >}{< f_1, f_1 >} f_1 + \frac{< v, f_2 >}{< f_2, f_2 >} f_2 = \frac{1}{2}(1,0,0,1) = (1/2,0,0,1/2).$$

The distance from v to W is $||v - P_W(v)|| = ||(-1/2,0,0,1/2)| = 1/\sqrt{2}$.

4. (a) If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ we have

$$\begin{aligned} < T(X), Y > &= \mathrm{Tr}((AX - XA)^t Y) = \mathrm{Tr}(X^t A Y - AX^t Y) = \mathrm{Tr}(X^t A Y) - \mathrm{Tr}(AX^t Y) \\ &= \mathrm{Tr}(X^t A Y) - \mathrm{Tr}(X^t Y A) = \mathrm{Tr}(X^t A Y - X^t Y A) =< X, T(Y) > . \end{aligned}$$

Hence T is self-adjoint. This can also be shown by showing that the matrix of T with respect to the standard basis is a symmetric matrix. This is due to the fact that the standard basis is orthonormal for the given inner product. To prove the second asssertion, we have

$$T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} c-b & d-a \\ b-c & a-d \end{bmatrix}, \quad T^2(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} 2(a-d) & 2(b-c) \\ 2(c-b) & 2(d-a) \end{bmatrix}, \quad T^3(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = 4T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}).$$

(b) The possible eigenvalues of T are 0 and ± 2.

$$\mathrm{Ker(T)} = \mathrm{Span}(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$$
$$\mathrm{Ker(T-2)} = \mathrm{Span}(\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}), \quad \mathrm{ker(T+2)} = \mathrm{Span}(\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}).$$

The required orthonormal basis is

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.$$

5. (a)

$$< T(f), g >= \int_{-1}^{1} \frac{d}{dx}((1-x^2)f'(x))g(x)dx = \int_{-1}^{1} (1-x^2)f'(x)g'(x) =< f, T(g) > .$$

(b) Since $T(x^k) = -k(k+1)x^k + k(k-1)x^{k-2}$ the subspace W_n is T-invariant. Moreover, the matrix of T with respect to the basis $1, x, \ldots, x^{k-1}$ is upper triangular with the k-th diagonal entry equal to $-k(k+1)$. Since these diagonal entries are the eigenvalues of the reatriction of T to W and they are distinct, it follows that the eigenspapces of the restriction are one-dimensional. Hence there is, up to multiplication by a non-zero scalar, a unique polynomial f_k such that $T(f_k) = -k(k+1)f_k$. Moreover, this polynomial must be of degree k.

(c) This follows by induction on n. For $n = 0$ we have $p_0 = 1$ and $T(p_0) = 0$. If the result is true for $n < k$ we use the fact that the orthogonal complement of W_k in W_{k+1} is $\mathbb{R}p_k$ and the fact that it is T-invatriant.

(d) $q_n = \frac{(2n)!}{n!}x^n +$ terms of lower degree and integration by parts shows that $q_n \in W_n^{\perp}$. The result follows since p_n is monic.

(e) Using the above one finds

$$p_0 = 1, \ p_1 = x, \ p_2 = x^2 - \frac{1}{3}, \ p_3 = x^3 - \frac{3}{5}x, \ p_4 = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$$

with corresponding eigenvalues $0, -2, -6, -12, -20$.

(f) This has been shown above.