McGill University
Math 247B: Linear Algebra
Solution Sheet for Assignment 6

1. If $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ -5 & -3 \end{bmatrix}$ the system is $\frac{dX}{dt} = AX$ whose solution is $X = e^{At}X(0)$. The characteristic matrix of A is $\lambda^2 + 2\lambda + 2$ which has the distinct complex roots $-1+i$, $-1-i$ with corresponding eigenvectors $\begin{bmatrix} 1 \\ -2+i \end{bmatrix}$, $\begin{bmatrix} 1 \\ -2-i \end{bmatrix}$. We then have

$$\begin{aligned} X(0) &= \frac{1-3i}{2}\begin{bmatrix} 1 \\ -2+i \end{bmatrix} + \frac{1+3i}{2}\begin{bmatrix} 1 \\ -2-i \end{bmatrix} \\ X(t) &= \frac{1-3i}{2}e^{(-1+i)t}\begin{bmatrix} 1 \\ -2+i \end{bmatrix} + \frac{1+3i}{2}e^{(-1-i)t}\begin{bmatrix} 1 \\ -2-i \end{bmatrix} \\ &= e^{-t}\Re((1-3i)(\cos t + i\sin t)\begin{bmatrix} 1 \\ -2+i \end{bmatrix}) \\ &= \begin{bmatrix} e^{-t}\cos t + 3e^{-t}\sin t \\ e^{-t}\cos t - 7e^{-t}\sin t \end{bmatrix}. \end{aligned}$$

2. We have $(D-1)(D-2)^2(y) = x^2$ so that $D^3(D-1)(D-2)^2(y) = 0$. Hence

$$\begin{aligned} y \in \mathrm{Ker}(\mathrm{D^3(D-1)(D-2)^2}) &= \mathrm{Ker(D^3) \oplus Ker(D-1) \oplus Ker((D-2)^2)} \\ &= \mathrm{Span(1, x, x^2) \oplus Span(e^x) \oplus Span(e^{2x}, xe^{2x})} \\ &= \mathrm{Span(1, x, x^2, e^x, e^{2x}, xe^{2x})}. \end{aligned}$$

Thus there are scalars A, B, C, D, E, F such that $y = A + Bx + Cx^2 + De^x + Ee^{2x} + Fxe^{2x}$. Substituting this expression for y in the original equation, we get

$$(D^3 - 5D^2 + 8D - 4)(y) = -4Cx^2 + (16C - 4B)x + B - 4A - 10C$$

which is equal to x^3 iff $A = -11/8, B = -1, C = -1/4$. Hence the solution of the given diiferential equation is

$$y = -\frac{11}{8} - x - \frac{1}{4}x^2 + De^x + Ee^{2x} + Fxe^{2x}$$

with D, E, F arbitrary constants.

3. We have $s_{n+1} - s_n = ((n+2)(2^{n+1} + 3^{n+1}$ so that $(L-1)(s) = 4(2^n) + 2(n2^n) + 6(3^n) + 3(n2^n) \in \mathrm{Ker((L-2)^2(L-3)^2)}$. Hence

$$\begin{aligned} s \in \mathrm{Ker((L-2)^2(L-3)^2(L-1))} &= \mathrm{Ker(L-1) \oplus Ker(((L-2)^2) \oplus Ker((L-3)^2)} \\ &= \mathrm{Span((1)) \oplus Span((2^n), (n2^n)) \oplus Span((3^n), (n3^n))} \\ &= \mathrm{Span((1), (2^n), (n2^n), (3^n), (n3^n))}. \end{aligned}$$

Hence there are scalars A, B, C, D, E such that $s = A(1) + B(2^n) + C(n2^n) + D(3^n) + E(n3^n)$. Since this implies that $(L-1)(s) = (B+2C)(2^n) + C(n2^n) + (2D+3E)(3^n) + 2E(n3^n)$ we obtain $B + 2C = 4, C = 2, 2D + 3E = 6, 2E = 3$. Using $A + B + D = 2$, we get$A = 5/4, B = 0, C = 2, D = 3/4, E = 3/2$. Hence

$$s_n = \frac{5}{4} + n2^{n+1} + \frac{1}{4}3^{n+1} + \frac{1}{2}n3^{n+1}.$$

4. If S, T are simultaneously diagonalizable we have $ST = TS$ since there then is a basis for which the matrix representations of T and S are both diagonal matrices.

   To prove the converse let $v \in \mathrm{Ker(T-a)}$. Then $T(v) = av$ so that $ST(v) = aS(v)$. If $ST = TS$ we then have $TS(v) = aS(v)$ so that $S(v) \in \mathrm{Ker(T-a)}$. Hence, if $ST = TS$, the eigenspaces of T are S-invariant. We now use the fact that the restriction R of a diagonalizable operator S to an S-invariant subspace is also diagonalizable. Indeed, since S is diagonalizable, its minimal polynomial $m_S(\lambda)$ is a product of distinct linear factors and $m_S(R) = 0$ implies $m_R(\lambda) \mid m_S(\lambda)$ which in turn implies R diagonalizable since $m_R(\lambda)$ would be a product of distinct linear factors. Hence, each eigenspace of T has a basis of eignvectors of R which are also eigenvectors of S. But these basis vectors are then eigenvectors for both S and T. Since V is the direct sum of the eigenspaces of T we obtain a basis for V consisting of vectors which are eigenvectors for both S and T.