McGill University
Math 247B: Linear Algebra
Solution Sheet for Assignment 5

1. The vectors

$$f_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}, \quad f_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad f_2 = \begin{bmatrix}-1\\1\\1\end{bmatrix}$$

form a basis for $V = \mathbb{R}^{3\times1}$ and

$$e_1 = \frac{1}{4}f_1 + \frac{1}{4}f_2 - \frac{1}{2}f_3, \quad e_2 = \frac{1}{2}f_1 - \frac{1}{2}f_2, \quad e_3 = -\frac{1}{4}f_1 + \frac{3}{4}f_2 + \frac{1}{2}f_3,$$

where e_1, e_2, e_3 is the standard basis of V. Let T be the linear operator on V such that

$$f(f_1) = 2f_1, \quad T(f_2) = 2f_2, \quad T(f_3) = 3f_3.$$

The required matrix A is the matrix of T with respect to the basis e_1, e_2, e_3. Since

$$\begin{aligned} T(e_1) &= \frac{1}{4}T(f_1) + \frac{1}{4}T(f_2) - \frac{1}{2}T(f_3) = \frac{1}{2}f_1 + \frac{1}{2}f_2 - \frac{3}{2}f_3 \\ T(e_2) &= \frac{1}{2}T(f_1) - \frac{1}{2}T(f_2) = 2e_1 \\ T(e_3) &- \frac{1}{4}T(f_1) + \frac{3}{4}T(f_2) + \frac{1}{2}T(f_3) = -\frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{5}{2}e_3, \end{aligned}$$

we see that

$$A = [T]_e = \begin{bmatrix} \frac{5}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{5}{2} \end{bmatrix}.$$

The minimal polynomial of A is the minimal polynomial of T which is $\lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$ since T is diagonalizable. The characteristic polynomial of A is $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = (\lambda-2)^2(\lambda-3)$ since the geometric multiplicities for the eigenvalues 2 and 3 are respectively 2 and 1.

2. Let $A = \begin{bmatrix}1 & 2\\4 & 3\end{bmatrix}$. Then $T^2 - 4T - 5I)(X) = (A^2 - 4A - 5I)(X) = 0$ since $A^2 - 4A - 5I = 0$. Since

$$T(\begin{bmatrix}a & b\\c & d\end{bmatrix}) = \begin{bmatrix}a+2c & b+2d\\4a+c & 4b+d\end{bmatrix},$$

we see that $T \neq 5I$, $T \neq -I$ so that $\lambda^2 - 4\lambda - 5 = (\lambda-5)(\lambda+1)$ is the minimal polynomial of T. Hence T is diagonalizable with eigenvalues -1 and 5. We have

$$\begin{aligned} \mathrm{Ker(T+1)} &= \{X \mid \begin{bmatrix}2 & 2\\4 & 4\end{bmatrix} X = 0\} = \mathrm{Span}(\begin{bmatrix}1 & 0\\-1 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\0 & -1\end{bmatrix}) \\ \mathrm{Ker(T-5)} &= \{X \mid \begin{bmatrix}-4 & 2\\4 & -2\end{bmatrix} X = 0\} = \mathrm{Span}(\begin{bmatrix}1 & 0\\2 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 2\end{bmatrix}) \end{aligned}$$

so that $V = \mathrm{Span}(\begin{bmatrix}2 & 2\\4 & 4\end{bmatrix}, \begin{bmatrix}1 & 0\\-1 & 0\end{bmatrix}, \begin{bmatrix}-4 & 2\\4 & -2\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 2\end{bmatrix})$.

3. Since $A^2 = 100I$ and $A \neq \pm 10I$, the minimal polynomial of A is $\lambda^2 - 100 = (\lambda - 1)(\lambda + 10)$. The eigenvalues of A are thus $10, -10$ and the corresponding eigenspaces are respectively

$$\text{Nullspace}(A - 10) = \text{Span}(\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}), \quad \text{Nullspace(A + 10I)} = \text{Span}(\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ -4 \end{bmatrix}).$$

The matrix P whose columns are the above eigenvectors is

$$P = \begin{bmatrix} -1 & -2 & 4 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -4 \end{bmatrix}$$

and $P^{-1}AP = \text{diag}(10, 10, 10, -10, -10)$.

4. If $X_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ the system can be written $X_{n+1} = AX_n$ where $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. Its solution is $X_n = A^n X_0$. The characteristic polynomial of A is $\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$. The vector $f_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 3 and $f_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 2. Since

$$X_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

we see that $X_n = A^n X_0 = -2^n f_1 + 3^{n+1} f_2$ so that $x_n = -2^{n+1} + 3^{n+1}$ and $y_n = -2^n + 3^{n+1}$. It follows that x_n/y_n converges to 1 as $n \to \infty$.

5. The characteristic matrix of A is $\lambda^2 - 13\lambda + 36 = (\lambda - 4)(\lambda - 9)$. The vector $f_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue 4 and $f_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is an eigenvector with eigenvalue 9. If P_1, P_2 are the projections onto the eigenspaces $\mathbb{R}f_1$ and $\mathbb{R}f_2$ respectively, we have $A = 4Q_1 + 9Q_2$ with $Q_i = [P_i]_e$. Since $e_1 = \frac{4}{5}f_1 + \frac{1}{5}f_2,\ e_2 = -\frac{1}{5}f_1 + \frac{1}{5}f_2$ we have

$$P_1(e_1) = \frac{4}{5}f_1 = \frac{4}{5}e_1 - \frac{4}{5}e_2,\ P_1(e_2) = -\frac{1}{5}f_1 = -\frac{1}{5}e_1 + \frac{1}{5}e_2$$

$$P_2(e_1) = \frac{1}{5}f_2 = \frac{1}{5}e_1 + \frac{4}{5}e_2,\ P_2(e_2) = \frac{1}{5}f_2 = \frac{1}{5}e_1 + \frac{4}{5}e_2.$$

Since $I = Q_1 + Q_2$, $AQ_1 = 4Q_1$, $AQ_2 = 9Q_2$ the spectral decomposition of A is

$$A = 4\begin{bmatrix} \frac{4}{5} & -\frac{1}{5} \\ -\frac{4}{5} & \frac{1}{5} \end{bmatrix} + 9\begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{4}{5} & \frac{4}{5} \end{bmatrix}.$$

Since $Q_i^2 = Q_i$ and $Q_1Q_2 = Q_2Q_1 = 0$ it follows that

$$A^n = 4^n\begin{bmatrix} \frac{4}{5} & -\frac{1}{5} \\ -\frac{4}{5} & \frac{1}{5} \end{bmatrix} + 9^n\begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{4}{5} & \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{4^{n+1}+9^n}{5} & \frac{9^n-4^n}{5} \\ \frac{4\cdot 9^n-4^{n+1}}{5} & \frac{4\cdot 9^n+4^{n+1}}{5} \end{bmatrix},$$

$$B = 2\begin{bmatrix} \frac{4}{5} & -\frac{1}{5} \\ -\frac{4}{5} & \frac{1}{5} \end{bmatrix} + 3\begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{4}{5} & \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{11}{5} & \frac{1}{5} \\ \frac{4}{5} & \frac{14}{5} \end{bmatrix}.$$