McGill University
Math 247B: Linear Algebra
Solution Sheet for Assignment 4

1. (a) Let a, b, c be scalars with $a\sin(x) + b\sin(2x) + c\sin(3x) = 0$ for all $x \in \mathbb{R}$. Setting $x = \pi/2$, we get $a - c = 0$. Differentiating both sides of the above dependence relation, we get $a\cos(x) + 2b\cos(2x) + 2c\cos(3x) = 0$ for all x. Setting $x = 0, \pi/2$, we get $a + 2b + 3c = 0,\ -2b = 0$. Hence a, b, c satisfy $a - c = 0 = 0,\ b = 0,\ a + 3c = 0$ which implies that $a = b = c = 0$ and hence that $\sin(x), \sin(2x), \sin(3x)$ are linearly independent.

   (b) The given functions are linearly dependent since the four given functions are in $\mathrm{Span}(1, \mathrm{x}, \mathrm{x}^2)$, a subspace spanned by three functions.

   (c) The given functions are linearly dependent since $\sin^2(x) - \cos^2(x) + \cos(2x) = 0$ for all x.

2. (a) We have $a\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix} + b \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix} + c \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix} + d \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix} = 0$ iff

$$a + b + 2c + d = 0,\ -5a + b - 4c - 7d = 0,\ -4a - b - 5c - 5d = 0,\ 2a + 5b + 7c + d = 0$$

which is equivalent to $a + b + 2c + d = 0,\ 6b + 6c - d = 0$. Since there are solutions with $c = 1, d = 0$ and $c = 0, d = 1$ we see the the first two of the given matrices span V and hence are a basis for V since they are also linearly independent. Since the matrices

$$\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix},\ \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix},\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

span $\mathbb{R}^{2\times 2}$ they must be linearly independent and hence a basis for V.

   (b) Since $X = a\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} a+b & -5a+b \\ -4a-b & 2a+5b \end{bmatrix}$ we see that a matrix X in V has trace zero iff $3a + 6b = 0$ iff $a = -2b$ iff $X = b\begin{bmatrix} -1 & 11 \\ 7 & 1 \end{bmatrix}$. Hence $\begin{bmatrix} -1 & 11 \\ 7 & 1 \end{bmatrix}$ is the required basis.

3. (a) If $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ we have $T(aX + bY) = A(aX + bY) - (aX + bY)A = aAX = bAY - aXA - bYA = a(AX - XA) + b(AY - YA) = aT(X) + bT(Y)$. Hence T is linear. If $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have

$$T(X) = \begin{bmatrix} -4b + 2c & -2a - 2b + 2d \\ 4a + 2c - 4d & 4b - 2c \end{bmatrix} = a\begin{bmatrix} 0 & -2 \\ 4 & 0 \end{bmatrix} + b\begin{bmatrix} -4 & -2 \\ 0 & 4 \end{bmatrix} + c\begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix} + d\begin{bmatrix} 0 & 2 \\ -4 & 0 \end{bmatrix}.$$

Hence $T(X) = 0$ iff $-4b + 2c = -2a - 2b + 2d = 4a + 2c - 4d = 4b - 2c = 0$ iff $a + b - d = 2b - c = 0$ iff $a = -c/2 + d,\ b = c/2$. Hence $\begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a basis for Ker(T). We also have

$$\mathrm{Im(T)} = \mathrm{Span}(\begin{bmatrix} 0 & -2 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} -4 & -2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -4 & 0 \end{bmatrix}) = \mathrm{Span}(\begin{bmatrix} 0 & -2 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} -4 & -2 \\ 0 & 4 \end{bmatrix})$$

which shows that $\begin{bmatrix} 0 & -2 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} -4 & -2 \\ 0 & 4 \end{bmatrix}$ is a basis for Im(T) since these two matrices are also linearly independent.

   (b) Since $T(e_1) = -2e_2 + 4e_3,\ T(e_2) = -4e_1 - 2e_2 + 4e_4,\ T(e_3) = 2e_1 + 2e_3 - 2e_4,\ T(e_4) = 2e_2 - 4e_3$, we have

$$A = [T]_e = [T]_{e,e} = \begin{bmatrix} 0 & -4 & 2 & 0 \\ -2 & -2 & 0 & 2 \\ 4 & 0 & 2 & -4 \\ 0 & 4 & -2 & 0 \end{bmatrix}.$$

Since $T(f_1)=0,\ T(f_2)=-2e_1-2e_2+2e_3+2e_4,\ T(f_3)=-2e_1-4e_2+6e_3+2e_4,\ T(f_4)=2e_1+2e_2-2e_3-2e_4)$ we have

$$C=[T]_{f,e}=\begin{bmatrix}0&-2&-2&2\\0&-2&-4&2\\0&2&6&-2\\0&2&2&-2\end{bmatrix}.$$

Since $f_1=e_1+e_4,\ f_2=e_2+e_3,\ f_3=e_1+e_2+e_3,\ f_4=e_3+e_4$ we have

$$P=[I]_{f,e}=\begin{bmatrix}1&0&1&0\\0&1&1&0\\0&1&1&1\\1&0&0&1\end{bmatrix}.$$

Since $e_1=-f_2+f_3,\ e_2=f_1+2f_2-f_3-f_4,\ e_3=-f_1-f_2+f_3+f_4,\ e_4=f_1+f_2-f_3$ we have

$$Q=[I]_{e,f}=\begin{bmatrix}0&1&-1&1\\-1&2&-1&1\\1&-1&1&-1\\0&-1&1&0\end{bmatrix}.$$

Since $T(e_1)=-6f_1-8f_2+6f_3+6f_4,\ T(e_2)=2f_1+4f_2-6f_3+2f_4,\ T(e_3)=-4f_1-6f_2+6f_3+2f_4$ we have

$$B=[T]_{e,f}=\begin{bmatrix}-6&2&-4&6\\-8&4&-6&8\\6&-6&6&-6\\6&2&2&-6\end{bmatrix}.$$

Since $T(f_1)=0,\ T(f_2)=-2f_1-2f_2+4f_4,\ T(f_3)=-8f_1-10f_2+6f_3+10f_4,\ T(f_4)=2f_1+2f_2-4f_4$ we have

$$D=[T]_f=[T]_{f,f}=\begin{bmatrix}0&-2&-8&2\\0&-2&-10&2\\0&0&6&0\\0&4&10&-4\end{bmatrix}.$$

We have $A=[T]_e=[IT]_e=[I]_{f,e}[T]_{e,f}=PB$, $A=[T]_e=[TI]_e=[T]_{f,e}[I]_{e,f}=CQ$, $A=[T]_e=[ITI]_e=[I]_{f,e}[T]_f[I]_{e,f}=PDQ$, $B=[T]_{e,f}=[T]_f[I]_{e,f}=DQ$, $C=[T]_{f,e}=[I]_{f,e}[T]_f=PD$, $I=[I]_e=[I]_{f,e}[I]_{e,f}=PQ$, $I=[I]_f=[I]_{e,f}[I]_{f,e}=QP$.

4. (a) V is a subspace of $C^\infty(\mathbb{R})$ since $V=\mathrm{Ker}(\mathrm{D}^4)$. A basis for V is $1,x,x^2,x^3$ since they are linearly independent an span V.

   (b) T is linear since $T(af+bg)(x)=(af+bg)(x)+(af+bg)(1-x)=af(x)+bg(x)+af(1-x)+bg(1-x)=a(f(x)+f(1-x))+b(g(x)+g(1-x))=aT(f)(x)+bT(g)(x)==(aT(f)+bT(g)(x)$. Since $T^2(f)(x)=T(T(f))(x)=T(f)(x)+T(f)(1-x)=f(x)+f(1-x)+f(1-x)+f(1-(1-x))=2(f(x)+f(1-x))=2T(f)(x)$ we see that $T^2=2T$. If $f\in V$ we have $f(x)=a+bx+cx^2+dx^3$ so that $f(1-x)=a+b(1-x)+c(1-x)^2+d(1-x)^3=a+b+c+d+(-b-2c-3d)x+(c+3d)x^2-dx^3$. This implies that $T(f)\in V$ and hence that $T(V)\subseteq V$.

   (c) If $f_1(x)=1, f_2(x)=x, f_3(x)=x_2, f_4(x)=x^3$ then $f=(f_1,f_2,f_3,f_4)$ is a basis for V. Since $S(f_1)=2f_1$, $S(f_2)=f_1,\ S(f_3)=f_1-2f_2+2f_3, S(f_4)=f_1-3f_2+3f_2$ we have

$$A=[S]_f=\begin{bmatrix}2&1&1&1\\0&0&-2&-3\\0&0&2&3\\0&0&0&0\end{bmatrix}.$$

We have $A^2=[S]_f^2=[S^2]_f=[2S]_f=2[S]_f=2A$ since $T^2=2T$ implies $S^2=2S$.