McGill University
Math 247B: Linear Algebra
Solution Sheet for Assignment 3

1. The following generating sets are all linearly independent and hence bases for the subspace they span.

$$\begin{aligned}
U_1 &= \text{Span}((-2,1,0,0),(-1,0,1,0),(1,0,0,1))\\
U_2 &= \text{Span}((1,1,0,0),(0,0,1,1))\\
U_3 &= \text{Span}((1,1,1,1),\ (1,0,1,0),\ (1,1,0,0))\\
U_4 &= \text{Span}((1,1,0,1),\ (1,2,2,1))\\
U_1+U_2 &= \text{Span}((-2,1,0,0),(-1,0,1,0),(1,0,0,1),(1,1,0,0))\\
U_1+U_3 &= \text{Span}((-2,1,0,0),(-1,0,1,0),(1,0,0,1),(1,1,0,0))\\
U_1+U_4 &= \text{Span}((-2,1,0,0),(-1,0,1,0),(1,0,0,1),(1,1,0,1))\\
U_2+U_3 &= \text{Span}((1,1,0,0),(0,0,1,1),(1,0,1,0))\\
U_2+U_4 &= \text{Span}((1,1,0,0),(0,0,1,1),(1,1,0,1),(1,2,2,1))\\
U_3+U_4 &= \text{Span}((1,1,1,1),(1,0,1,0),(1,1,0,0),(1,1,0,1))\\
U_1\cap U_2 &= \text{Span}((0,0,1,1))\\
U_1\cap U_3 &= \text{Span}((1,-2,3,0),(-1,2,0,3))\\
U_1\cap U_4 &= \text{Span}((2,1,-2,2))\\
U_2\cap U_3 &= \text{Span}((1,1,0,0),(0,0,1,1))\\
U_2\cap U_4 &= \text{Span}(\emptyset)\\
U_3\cap U_4 &= \text{Span}((3,4,2,3))
\end{aligned}$$

2. We are looking for a linear operator T on $\mathbb{R}^4$ with $\text{Ker(T)} = \text{Im(T)} = \text{Span}((1,0,1,0),(1,1,1,1))$. If A is the matrix of T then the matrix A is the coefficient matrix of a homogeneous system of equations whose solution space is $\text{Span}((1,0,1,0),(1,1,1,1))$. Hence, if (a,b,c,d) is a row of A we must have $a+c=0, a+b+c+d=0$. Hence, $(a,b,c,d) = a(1,0,-1,0)+(0,1,0,-1)$. Since the columns of A must be linear combinations of the vectors $[1,0,1,0]^t, [0,1,0,1]^t$ the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1 \end{bmatrix}$$

yields an operator T with the required properties.

Another way of finding T is to complete $u_1 = (1,0,1,0), u_2 = (1,1,1,1)$ to a basis u_1,u_2,u_3,u_4 of $\mathbb{R}^4$. The vectors $u_3 = (1,0,0,0), u_4 = (0,1,0,0)$ will do the job. We want $T(u_1) = T(u_2) = 0,\ T(u_3) = u_1, T(u_4) = u_2$. Such a T is given by $T(x_1u_1 + x_2u_2 + x_2u_3 + x_4u_4) = x_3u_1 + x_4u_4$. If e_1,e_2,e_3,e_4 is the standard basis of $\mathbb{R}^4$ we have $e_1 = u_3, e_2 = u_4, e_3 = u_1 - u_3, e_4 = u_2 - u_1 - u_4$. Hence $T(e_1) = u_1,\ T(e_2) = u_2,\ T(e_3) = -u_1,\ T(e_4) = -u_2$ which shows that the matrix of T is the matrix A given above.

3. (a) M_g is linear since $(M_g(a_1f_1+a_2f_2))(x) = g(x)(a_1f_1(x)+a_2g(x)f_2(x)) = a_1g(x)f_1(x)+a_2g(x)f_2(x) = a_1M_g(f_1)(x)+a_2M_g(f_2)(x) = (a_1M_g(f_1)+a_2M_g(f_2))(x)$ which shows that $M_g(a_1f_1+a_2f_2) = a_1M_g(f_1)+a_2M_g(f_2)$.

   (b) $((DM_g - M_gD)(f))(x) = \frac{d}{dx}(xf(x)) - xf'(x) = f(x)+xf'(x)-xf'(x) = f(x)$. Hence $(DM_g - M_gD)(f) = f$ which gives $DM_g - M_gD = I$.

(c) If $g(a) = 0$ for some a then $M_g(f)(a) = g)a)f(a) = 0$ so that M_g cannot be onto since the constant function $h(x) = 1$ is not in the image of M_g. If $g(x) \neq 0$ for all x and $h(x) = 1/g(x)$, then $h = 1/g \in V$ and $M_h M_g = M_g M_h = I$ which shows that $M_g^{-1} = M_h = M_{1/g}$.

(d) $(M_g D M_g^{-1}(f))(x) = e^{ax}\frac{d}{dx}(e^{-ax}f(x)) = e^{ax}(e^{-ax}f'(x) - ae^{-ax}f(x)) = f'(x) - af(x) = ((D-a)(f))(x)$.

(e) $f \in \mathrm{Ker((D-a)(D-b))} \iff \mathrm{(D-a)(D-b)(f) = 0}$. But, using $D - a = M_g D M_g^{-1}$ where $g(x) = e^{ax}$, we see that $(D-a)(D-b)(f) = 0 \iff M_g D M_g^{-1}(D-b)(f) = 0 \iff D M_g^{-1}(D-b)(f) = 0 \iff g^{-ax}(f'(x) - bf(x)) = C$, ($C$ an arbitary constant). Hence $f \in \mathrm{Ker((D-a)(D-b))} \iff \mathrm{f'(x) - bf(x) = Ce^{ax}} \iff \mathrm{((D-b)(f))(x) = Ce^{ax}}$. Using $D - b = M_h D M_h^{-1}$ with $h(x) = e^{bx}$, we see that $f'(x) - bf(x) = Ce^{ax} \iff \frac{d}{dx}(e^{-bx}f(x) = Ce^{(a-b)x}$. If $a \neq b$ the latter holds iff $e^{-bx}f(x) = C_1 e^{(a-b)x} + C_2$ with $C_1 = C/(a-b), C_2 \in \mathbb{R}$. Hence, if $a \neq b$, we have $f \in \mathrm{Ker((D-a)(D-b))} \iff \mathrm{f(x) = C_1e^{ax} + C_2e^{bx}}$ which shows that $\mathrm{Ker((D-a)(D-b)) = Ker(D-a) + Ker(D-b)}$. If $a = b$ we get $f \in \mathrm{Ker((D-a)^2)} \iff \mathrm{f(x) = C_1xe^{ax} + C_2e^{ax}}$.

4. (a) The zero sequence is in W since $p_0(n)x_{n+k} + p_1(n)x_{n+k-1} + \ldots + p_k(n)x_n = 0$ if $x_i = 0$ for all i. If $x, y \in W$ and $a, b \in F$ we have

$$p_0(n)(ax+by)_{n+k} + p_1(n)(ax+by)_{n+k-1} + \ldots + p_k(n)(ax+by)_n =$$

$$a(p_0(n)x_{n+k} + p_1(n)x_{n+k-1} + \ldots p_k(n)x_n) + b(p_0(n)y_{n+k} + p_1(n)y_{n+k-1} + \ldots + p_k(n)y_n) = 0$$

which implies that $ax + by \in W$.

(b) T is linear since $T(ax+by) = (ax_0 + by_0, \ldots, ax_{k-1} + by_{k-1}) = a(x_0, \ldots, x_{k-1}) + b(y_0, \ldots, y_{k-1}) = aT(x) + bT(y)$. If $p_0(n) \neq 0$ for all n we have $x_{n+k} = q_1(n)x_{n+k-1} + \ldots + q_k(n)x_n = 0$ with $q_i(n) = -p_i(n)/p_0(n)$. Hence, if $x \in W$ we have $x = 0$ if $x_0 = x_1 = \cdots = x_{k-1} = 0$. This shows that T is one-to-one. Finally, T is onto since, given $x_0, \ldots, x_{k-1} \in F$ we can use the formula for x_{n+k} to inductively define x_n for $n \geq k$. We have then $x \in W$ and $T(x) = (x_0, \ldots, x_{k-1})$. If $p_0(n) = 0$ for some n then T need not be an isomorphism. For example, for the recurrence equation $na_{n+1} - a_n = 0 \quad (n \geq 0)$ we have $a_0 = 0$ so that $T : W \to \mathbb{R}$ is the zero mapping.