McGill University
Math 247B: Linear Algebra
Solution Sheet for Assignment 2

1. (a) $0 \in W$ since $0_n = 0$ for all $n \implies 0_{n+3} = 0_{n+2} + 0_n = 0$ for all n. If $x, y \in W$, $a, b \in F$ we have $(ax+by)_{n+3} = ax_{n+3} + by_{n+3} = a(x_{n+2} + x_n) + b(y_{n+2} + y_n) = ax_{n+2} + by_{n+2} + ax_n + by_n = (ax+by)_{n+2} + (ax+by)_n$ which implies $ax + by \in W$. Hence W is a subspace of V.

   (b) $0 \in W$ since $0(x) = 0$ for all x implies $0(x) = 0(x^2) = 0$. If $f, g \in W$, $a, b \in \mathbb{R}$ we have $(af+bg)(x) = af(x) + bg(x) = af(x^2) + bg(x^2)$ which implies $af + bg \in W$. Hence W is a subspace of V.

   (c) Let $f, g \in V$ be defined by $f(x) = x$, $g(x) = 1 - x$. Then $f, g \in W$ since $f(0) = g(1) = 0$ but $f + g \notin W$ since $(f+g)(0) = (f+g)(1) = 1$. Hence W is not a subspace of V.

   (d) Since $x_1^2 + x_1x_2 + x_2^2 = (x_1 + x_2/2)^2 + 3x_2^2/4$ we see that $x_1^2 + x_1x_2 + x_2^2 = 0$ iff $x_1 = x_2 = 0$. Hence $W = \mathrm{Span}((0,0,1))$, a subspace of V.

   (e) $0 \in W$ since $0^t = 0 = -0$. If $X, Y \in W$, $a, b \in F$ we have $(aX + bY)^t = aX^t + bY^t = -aX - bY = -(aX + bY)$ which implies $aX + bY \in W$. Hence W is a subspace of V.

2. $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathrm{Span}((1,1,0,1,1,1), (1,0,1,1,1,0), (1,0,0,1,1,0), (1,1,1,1,1,1))$ iff there are scalars c_1, c_2, c_3, c_4 such that $x = c_1(1,1,0,1,1,1) + c_2(1,0,1,1,1,0) + c_3(1,0,0,1,1,0) + c_4(1,1,1,1,1,1)$. But this holds iff the system of equations

$$
\begin{aligned}
c_1 + c_2 + c_3 + c_4 &= x_1 \\
c_1 + c + 4 &= x_2 \\
c_2 + c_4 &= x_3 \\
c_1 + c_2 + c_3 + c_4 &= x_4 \\
c_1 + c_2 + c_3 + c_4 &= x_5 \\
c_1 + c_4 &= x_6
\end{aligned}
$$

has a solution for c_1, c_2, c_3, c_4. Bringing this system to echelon form we get

$$
\begin{aligned}
c_1 + c_2 + c_3 + c_4 &= x_1 \\
c_2 + c_3 &= x_1 + x_2 \\
c_3 + c_4 &= x_1 + x_2 + x_3 \\
0 &= x_1 + x_4 \\
0 &= x_1 + x_5 \\
0 &= x_2 + x_6
\end{aligned}
$$

which shows that the system has a solution iff $x_1 + x_4 = 0, x_1 + x_5 = 0, x_2 + x_6 = 0$.

3.

$$\begin{aligned}
U_1 &= \{(-2r-s+t,r,s,t) \mid r,s,t \in \mathbb{R}\} = \mathrm{Span}((-2,1,0,0),(-1,0,1,0),(1,0,0,1)) \\
U_2 &= \{(s,s,t,t) \mid s,t \in \mathbb{R}\} = \mathrm{Span}((1,1,0,0),(0,0,1,1)) \\
U_3 &= \mathrm{Span}((1,1,1,1),\ (1,0,1,0),\ (1,1,0,0)) = \{(\mathrm{x_1,x_2,x_3,x_4}) \mid \mathrm{x_1-x_2-x_3+x_4=0}\} \\
U_4 &= \mathrm{Span}((1,1,0,1),\ (1,2,2,1)) = \{(\mathrm{x_1,x_2,x_3,x_4}) \mid \mathrm{2x_1-2x_2+x_3=0,x_1-x_4=0}\} \\
U_1+U_2 &= \mathrm{Span}((-2,1,0,0),(-1,0,1,0),(1,0,0,1),(1,1,0,0),(0,0,1,1)) \\
U_1+U_3 &= \mathrm{Span}((-2,1,0,0),(-1,0,1,0),(1,0,0,1),(1,1,1,1),(1,0,1,0),(1,1,0,0)) \\
U_1+U_4 &= \mathrm{Span}((-2,1,0,0),(-1,0,1,0),(1,0,0,1),(1,1,0,1),(1,2,2,1)) \\
U_2+U_3 &= \mathrm{Span}((1,1,0,0),(0,0,1,1),(1,1,1,1),(1,0,1,0),(1,1,0,0)) \\
U_2+U_4 &= \mathrm{Span}((1,1,0,0),(0,0,1,1),(1,1,0,1),(1,2,2,1)) \\
U_3+U_4 &= \mathrm{Span}((1,1,1,1),(1,0,1,0),(1,1,0,0),(1,1,0,1),(1,2,2,1)) \\
U_1 \cap U_2 &= \{(x_1,x_2,x_3,x_4) \mid x_1+2x_2+x_3-x_4=0, x_1-x_2=0, x_3-x_4=0\} = \mathrm{Span}((0,0,1,1)) \\
U_1 \cap U_3 &= \{(x_1,x_2,x_3,x_4) \mid x_1+2x_2+x_3-x_4=0, x_1-x_2-x_3+x_4=0\} = \mathrm{Span}((1,-2,3,0),(-1,2,0,3)) \\
U_1 \cap U_4 &= \{(x_1,x_2,x_3,x_4) \mid x_1+2x_2+x_3-x_4=0, 2x_1-2x_2+x_3=0, x_1-x_4=0\} = \mathrm{Span}((2,1,-2,2)) \\
U_2 \cap U_3 &= \{(x_1,x_2,x_3,x_4) \mid x_1-x_2=0, x_3-x_4=0, x_1-x_2-x_3+x_4=0\} = U_2 \\
U_2 \cap U_4 &= \{(x_1,x_2,x_3,x_4) \mid x_1-x_2=0, x_3-x_4=0, 2x_1-2x_2+x_3=0, x_1-x_4=0\} = \mathrm{Span}((0,0,0,0)) \\
U_3 \cap U_4 &= \{(x_1,x_2,x_3,x_4) \mid x_1-x_2-x_3+x_4=0, 2x_1-2x_2+x_3=0, x_1-x_4=0\} = \mathrm{Span}((3,4,2,3))
\end{aligned}$$

4. (a) If $U \subseteq V$ then $U \cup V = V$, a subspace of W. Similarly, if $V \subseteq U$ then $U \cup V = U$, a subspace of W. Now suppose that $U \cup V$ is a subspace of W. If neither $U \subseteq V$ nor $V \subseteq U$ hold we can find $u \in U, u \notin V$ and $v \in V, v \notin U$. But then $u+v \notin U, V$ which contradicts the fact that $U \cup V$ is a subspace of W.

   (b) T is linear since $T(a_1(u_1,v_1)+a_2(u_2,v_2)) = T(a_1u_1+a_2u_2, a_1v_1+a_2u_2) = a_1u_1+a_2u_2+a_1v_1+a_2u_2 = a_1(u_1+v_1)+a_2(u_2+v_2) = a_1T(u_1,v_1)+a_2T(u_2,v_2)$. The image of T is $U+V$ so that T is onto iff $W = U+V$. We have $T(u,v)=0$ iff $u+v=0$ so that $\mathrm{Ker(T)} = \{(\mathrm{w},-\mathrm{w}) \mid \mathrm{w} \in \mathrm{U} \cap \mathrm{V}\}$. Hence T is one-to-one iff $U \cap V = \{0\}$. Consequently T is an isomorphism iff $U+V=W$ and $U \cap V = \{0\}$.