McGill University
Math 247B: Linear Algebra
Solution Sheet for Assignment 1

1. (a) $R(A \cup B) = R(A) \cup R(B)$ since $y \in R(A \cup B) \iff (\exists x \in A \cup B)\ (x,y) \in R \iff$ $(\exists x \in A)\ (x,y) \in R$ or $(\exists x \in B)\ (x,y) \in R \iff y \in R(A)$ or $y \in R(B) \iff y \in R(A) \cup R(B)$.

   (b) $R(A \cap B) \neq R(A) \cap R(B)$ in general since in the case $R = \{(1,1),(2,1)\},\ A = \{1\},\ B = \{2\}$ we have $R(A \cap B) = R(\emptyset) = \emptyset$ while $R(A) \cap R(B) = \{1\} \cap \{1\} = \{1\}$.

   (c) If R is a function we have $R^{-1}(A \cap B) = R^{-1}(A) \cap R^{-1}(B)$ since $y \in R^{-1}(A \cap B) \iff$ $(\exists x \in A \cap B)\ (x,y) \in R^{-1} \iff (\exists x \in A)\ (x,y) \in R^{-1}$ and $(\exists x' \in B)\ (x',y) \in R^{-1} \iff$ $y \in R^{-1}(A)$ and $y \in R^{-1}(B) \iff y \in R^{-1}(A) \cup R^{-1}(B)$. Note that $(x,y),(x',y) \in R^{-1}$ implies that $(y,x),(y,x') \in R$ and hence that $x = x'$ since R is a function.

2. (a) Since $F[\sqrt{\alpha}]$ contains F we only have to show that $F[\sqrt{\alpha}]$ is closed under addition, multiplication and division by non-zero elements. This follows from $(a + b\sqrt{\alpha}) + (a' + b'\sqrt{\alpha} = (a + a') + (b + b')\sqrt{\alpha}$, $-(a + b\sqrt{\alpha}) = (-a) + (-b)\sqrt{\alpha}$, $(a + b\sqrt{\alpha})(a' + b'\sqrt{\alpha} = (aa' + bb'\alpha) + (ab' + a'b)\sqrt{\alpha}$, $1/a + b\sqrt{\alpha} = a/(a^2 + b^2\alpha) + (-b/(a^2 + b^2)\alpha)\sqrt{\alpha}$. Note that $a + b\sqrt{\alpha} = 0 \implies b = 0$ since $\sqrt{\alpha} \notin F$ and hence that $a = 0$. Since $a + b\sqrt{\alpha} = a' + b\sqrt{\alpha} \iff (a - a') + (b - b')\sqrt{\alpha} = 0$ we see that every element of $F[\sqrt{\alpha}$ can be uniquely written in the form $a = b\sqrt{\alpha}$ with $a, b \in F$.

   (b) Since $u = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} = (a = b\sqrt{2} + (c + d\sqrt{2}\sqrt{3}$ we have $F = (\mathbb{Q}[\sqrt{2}])[\sqrt{3}]$. Note that $\sqrt{3} \notin \mathbb{Q}[\sqrt{2}]$. Since $\sqrt{3} = a + b\sqrt{2}$ with $a, b \in \mathbb{Q}$ implies that $3 = a^2 + 2b^2 + ab\sqrt{2}$ and hence that a or $b = 0$ in which case either $3 = 2b^2$ or $3 = a^2$, both of which are impossible. Hence F is a field by (a) and $u = 0 \iff a + b\sqrt{2} = c + d\sqrt{2} = 0 \iff a = b = c = d = 0$. Also

$$
\begin{aligned}
1/u &= (a + b\sqrt{2}) - (c + d\sqrt{2})\sqrt{3})((a + b\sqrt{2})^2 + 3(c + d\sqrt{2})^2)^{-1} \\
&= (a + b\sqrt{2} - c\sqrt{3} - d\sqrt{6})(a^2 + 2b^2 + 3c^2 + 3d^2) + (2ab + 6cd)\sqrt{2})^{-1} \\
&= (a + b\sqrt{2} - c\sqrt{3} - d\sqrt{6})((a^2 + 2b^2 + 3c^2 + 3d^2) - (2ab + 6cd)\sqrt{2})((a^2 + 2b^2 + 3c^2 + 3d^2)^2 + 2(2ab + 6cd)^2)^{-1}.
\end{aligned}
$$

3. Mutiplying the third equation by 1/2 and then interchanging the first and third equations, we get

$$
\begin{aligned}
x_1 - x_2 + x_3 + 2x_4 - 4x_5 &= 0 \\
2x_1 - 2x_2 + 2x_3 + 3x_4 - 5x_5 &= 0 \\
2x_1 - 3x_2 + 6x_3 + 2x_4 - 5x_5 &= 0 \\
5x_1 - 6x_2 + 9x_3 + 7x_4 - 14x_5 &= 0
\end{aligned}
$$

Using the first equation to eliminate x_1 from the other equations, we get

$$
\begin{aligned}
x_1 - x_2 + x_3 + 2x_4 - 4x_5 &= 0 \\
-x_4 + 3x_5 &= 0 \\
-x_2 + 4x_3 - 2x_4 + 3x_5 &= 0 \\
-x_2 + 4x_3 - 3x_4 + 6x_5 &= 0
\end{aligned}
$$

Now multiply equations 2 and 3 by -1 and then interchange equations 2 and 3 to get

$$
\begin{aligned}
x_1 - x_2 + x_3 + 2x_4 - 4x_5 &= 0 \\
x_2 - 4x_3 + 2x_4 - 3x_5 &= 0 \\
x_4 - 3x_5 &= 0 \\
-x_2 + 4x_3 - 3x_4 + 6x_5 &= 0
\end{aligned}
$$

Now add the second equation to the fourth to eliminate x_2

$$\begin{aligned} x_1 - x_2 + x_3 + 2x_4 - 4x_5 &= 0 \\ x_2 - 4x_3 + 2x_4 - 3x_5 &= 0 \\ x_4 - 3x_5 &= 0 \\ -x_4 + 3x_5 &= 0 \end{aligned}$$

Adding the third equation to the fourth and deleting the zero equation, we get

$$\begin{aligned} x_1 - x_2 + x_3 + 2x_4 - 4x_5 &= 0 \\ x_2 - 4x_3 + 2x_4 - 3x_5 &= 0 \\ x_4 - 3x_5 &= 0 \end{aligned}$$

By backsubstitution we get $x_4 = 3x_5$, $x_2 = 4x_3 - 2x_4 + 3x_5 = 4x_3 - 3x_5$, $x_1 = x_2 - x_3 - 2x_4 + 4x_5 = 3x_3 - 5x_5$. Hence the solution set is $\{(3s - 5t, 4s - 3t, s, 3t, t) \mid s, t \in \mathbb{R}\}$.

4. Using equation 1 to eliminate x_1 from the other equations, we get

$$\begin{aligned} x_1 + x_2 + x_3 + x_7 + x_8 &= 1 \\ x_3 + x_5 + x_8 + x_9 &= 0 \\ x_3 + x_6 &= 0 \\ x_2 + x_4 + x_5 + x_6 + x_7 + x_8 &= 0 \end{aligned}$$

Now interchange the second and fourth equations to get

$$\begin{aligned} x_1 + x_2 + x_3 + x_7 + x_8 &= 1 \\ x_2 + x_4 + x_5 + x_6 + x_7 + x_8 &= 0 \\ x_3 + x_6 &= 0 \\ x_3 + x_5 + x_8 + x_9 &= 0 \end{aligned}$$

Now add the third equation to the fourth to get

$$\begin{aligned} x_1 + x_2 + x_3 + x_7 + x_8 &= 1 \\ x_2 + x_4 + x_5 + x_6 + x_7 + x_8 &= 0 \\ x_3 + x_6 &= 0 \\ x_5 + x_6 + x_8 + x_9 &= 0 \end{aligned}$$

Adding equation 5 to equation 2 and then equation 3 to equation 1, we get

$$\begin{aligned} x_1 + x_2 + x_6 + x_7 + x_8 &= 1 \\ x_2 + x_4 + x_7 + x_9 &= 0 \\ x_3 + x_6 &= 0 \\ x_5 + x_6 + x_8 + x_9 &= 0 \end{aligned}$$

Finally add equation 2 to equation 1 to get

$$\begin{aligned} x_1 + x_4 + x_6 + x_8 + x_9 &= 1 \\ x_2 + x_4 + x_7 + x_9 &= 0 \\ x_3 + x_6 &= 0 \\ x_5 + x_6 + x_8 + x_9 &= 0 \end{aligned}$$

Solving for x_1, x_2, x_3, x_5 in terms of $x_4 = r, x_6 = s, x_7 = t, x_8 = u, x_9 = v$ we find the solution set to be

$$\{(1 + r + s + u + v, r + t + v, s, r, s + u + v, s, t, u, v) \mid r, s, t, u, v \in \mathbb{F}_2\}.$$

This set has $2^5 = 32$ elements.