McGill University
Math 247B: Linear Algebra
Midterm Test

Attempt all questions

1. Let $V = \mathbb{R}^{\mathbb{R}}$ be the vector space of real valued functions on the real line.

   (a) Show that $W_1 = \{f \in V \mid f(x) = f(x^2) \text{ for all } x \in \mathbb{R}\}$ is a subspace of V. Find a linear operator on V whose kernel is W_1.

   (b) Show that $W_2 = \{f \in V \mid f(x) = f(x)^2 \text{ for all } x \in \mathbb{R}\}$ is not a subspace of V by finding a function $f \in W_2$ with $2f \notin W_2$.

2. (a) Define the terms "linearly independent sequence of vectors" and "linearly dependent sequence of vectors".

   (b) Show that $e^x, e^{2x}, e^{3x} \in \mathbb{R}^{\mathbb{R}}$ is a linearly independent sequence of functions.

   (c) If A is a 2×2 matrix, prove that I, A, A^2, A^3, A^4 is a linearly dependent sequence of matrices. Quote any theorem that you use in your proof.

3. (a) If A is a fixed 2×2 matrix over a field F and $V = F^{2\times 2}$ is the vector spaces of 2×2 matrices over F, show that the mapping $T : V \to V$ defined by $T(X) = AX - XA$ is linear.

   (b) If $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$ in part (a), find bases for the kernel and image of T.

   (c) Quote a theorem relating the dimensions of the image and kernel of a linear mapping.