McGill University
Math 247B: Linear Algebra
Assignment 7: due Tuesday, April 11, 2000

1. If $A$ is the matrix
$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix},$$
find an invertible matrix $P$ such that $P^{-1}AP$ is upper triangular.

2. Repeat question 1 with
$$A = \begin{bmatrix} 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}.$$

3. Find the orthogonal projection of $v = (0, 0, 0, 1) \in \mathbb{R}^4$ on the subspace
$$W = \{(x_1, x_2, x_3, x_4) \mid x_1 - x_2 + x_3 - x_4 = x_1 - x_4 = 0\}.$$
Find the distance of $v$ to $W$.

4. Let $V = \mathbb{R}^{2\times 2}$ with the inner product $< X, Y >= Tr(X^tY)$ and let $T$ be the linear operator on $V$ defined by
$$T(X) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} X - X \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$
   (a) Show that $T$ is self-adjoint and that $T^3 = 4T$;
   (b) Find an orthormal basis of $V$ consisting of eigenvectors of $T$.

5. Let $V = C^\infty[-1, 1]$ with the inner product
$$< f, g >= \int_{-1}^{1} f(x)g(x)dx$$
and let $T$ be the linear operator of $V$ defined by
$$T(f(x)) = (1 - x^2)f''(x) - 2xf'(x)).$$
   (a) Show that $T$ is self-adjoint. Hint: Use the fact that $T(f(x)) = \frac{d}{dx}((1 - x^2)f'(x);$
   (b) Show that $W_n = \mathrm{Span}(1, \mathrm{x}, \ldots, \mathrm{x}^{\mathrm{n}-1})$ is $T$-invariant;
   (c) If $p_0, p_1, \ldots, p_{n-1}$ are the functions obtained from $1, x, \ldots, x^{n-1}, \ldots$ by the Gram-Schmidt process, show that for $n \geq 0$ the polynomial $p_n$ is an eigenfunction of $T$;
   (d) If $q_n = \frac{d^n}{dx^n}((x^2-1)^n)$, show that $p_n = \frac{n!}{(2n)!}q_n$. Hint: Show that $q_n \in W_{n+1}$ and $q_n \in W_n^\perp$.
   (e) Compute $p_1, p_2, p_3, p_4$ and their eigenvalues.
   (f) If $T(p_n) = \lambda_n p_n$, show that $\lambda_n = -n(n+1)$. Hint: Show that There is a unique non-zero polynomial of degree $n$ that $T(p_n) = -n(n+1)p_n$.
   The polynomial $p_n$ is, up to a constant factor, the Legendre polynomial of degree $n$.