McGill University
Math 247B: Linear Algebra
Assignment 4: due Friday, February 18, 2000

1. Which of the following sequences of functions in $C^\infty(\mathbb{R})$ are linearly independent over $\mathbb{R}$? Justify your answers.

   (a) $\sin(x),\ \sin(2x),\ \sin(3x)$;

   (b) $(x-1)^2,\ (x+1)^2,\ (x-2)^2,\ (x+2)^2$;

   (c) $\sin^2(x),\ \cos^2(x),\ \cos(2x)$.

2. Let V be the subspace of $\mathbb{R}^{2\times 2}$ generated by the matrices

$$\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix}, \quad \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix}.$$

   (a) Find a basis for V and complete this basis to a basis for $R^{2\times 2}$;

   (b) Find a basis for the subspace of V consisting of those matrices in V which have trace zero.

3. Let $V = \mathbb{R}^{2\times 2}$ and let $T : V \to V$ be the mapping defined by

$$T(X) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} X - X \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

   (a) Show that T is linear and find bases for the kernel and image of T;

   (b) Find $A = [T]_e,\ B = [T]_{e,f},\ C = [T]_{f,e}, D = [T]_f,\ P = [I]_{f,e}, Q = [I]_{e,f}$ where

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$f_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad f_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad f_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

   Find the relations that exist between A, B, C, D, P, Q.

4. Let $V = \{f \in \mathbb{R}^{[0,1]} \mid f(x) = a + bx + cx^2 + dx^3, a, b, c, d \in \mathbb{R}\}$ and let $T : \mathbb{R}^{[0,1]} \to \mathbb{R}^{[0,1]}$ be the mapping defined by $T(f)(x) = f(x) + f(1-x)$.

   (a) Show that V is a subspace of $\mathbb{R}^{[0,1]}$ and find a basis for V;

   (b) Show that T is linear, that $T^2 = 2T$ and that $T(V) \subseteq V$;

   (c) If S is the restriction of T to V, find the matrix of S with respect to some basis for V. If A is this matrix, show that $A^2 = 2A$.