McGill University
Math 247B: Linear Algebra
Assignment 3: due Monday, February 14, 2000

1. Find bases for each of the subspaces U_i, $U_i + U_j$, $U_i \cap U_j$ in question 3 of assignment 2.

2. Construct a linear operator on $\mathbb{R}^4$ whose kernel and image are are spanned by the vectors $(1, 0, 1, 0)$, $(1, 1, 1, 1)$.

3. Let $V = C^\infty(\mathbb{R})$ and let $D = \frac{d}{dx}$ be the differentiation operator on V. If $g \in V$, let $M_g : V \to V$ be defined by $M_g(f)(x) = g(x)f(x)$.

   (a) Show that M_g is a linear operator on V;

   (b) If $g(x) = x$ for all x, show that $DM_g - M_g D = I$ where I is the identity operator on V;

   (c) Show that M_g is bijective iff $g(x) \neq 0$ for all x, in which case, $M_g^{-1} = M_{1/g}$;

   (d) If $g(x) = e^{ax}$ ($a \in \mathbb{R}$ fixed), show that $D - a = M_g D M_g^{-1}$;

   (e) Using (d), show that $\mathrm{Ker((D-a)(D-b))} = \mathrm{Ker(D-a)} + \mathrm{Ker(D-b)}$ if $a \neq b$. What happens if $a = b$?

4. Let F be a field and let $p_0, p_1, \dots p_k \in F^{\mathbb{N}}$ ($k \geq 1$) and let

$$W = \{x \in F^{\mathbb{N}} \mid p_0(n)x_{n+k} + p_1(n)x_{n+k-1} + \dots p_k(n)x_n = 0, \quad n \geq 0\}.$$

   (a) Show that W is a subspace of $F^{\mathbb{N}}$;

   (b) Show that the mapping $T : W \to F^k$ defined by $T(x) = (x_0, x_1, \dots x_{k-1})$ is an isomorphism of vector spaces if $p_0(n) \neq 0$ for all $n \in \mathbb{N}$. What happens if $p_0(n) = 0$ for some n?