McGill University
Math 247B: Linear Algebra
Assignment 1: due Wednesday, January 26, 2000

1. Let A, B be sets and R a relation. Prove or disprove the following statements:
   (a) $R(A \cup B) = R(A) \cup R(B)$;
   (b) $R(A \cap B) = R(A) \cap R(B)$;
   (c) If R is a function then $R^{-1}(A \cap B) = R^{-1}(A) \cap R^{-1}(B)$.

2. (a) Let F be a subfield of $\mathbb{R}$ and let $\alpha \in F$ with $\alpha > 0$ and $\sqrt{\alpha} \notin F$. If
   $$F[\sqrt{\alpha}] = \{a + b\sqrt{\alpha} \mid a, b \in F\},$$
   show that $F[\sqrt{\alpha}]$ is a subfield of $\mathbb{R}$ which contains F and that every element of $F[\sqrt{\alpha}]$ can be uniquely writen in the form $a + b\sqrt{\alpha}$ with $a, b \in F$.
   (b) Show that $F = \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} \mid a, b, c, d \in \mathbb{Q}\}$ is a subfield of $\mathbb{R}$ by showing that $F = (\mathbb{Q}[\sqrt{2}])[\sqrt{3}]$. If $u = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$ with $a, b, c, d \in \mathbb{Q}$, show that $u = 0$ iff $a = b = c = d = 0$ and, if $u \neq 0$, express u^{-1} in the form $a_1 + b_1\sqrt{2} + c_1\sqrt{3} + d_1\sqrt{6}$ with $a_1, b_1, c_1, d_1 \in \mathbb{Q}$.

3. Using Gaussian elimination, solve the system of equations
   $$\begin{aligned} 2x_1 - 3x_2 + 6x_3 + 2x_4 - 5x_5 &= 0 \\ 2x_1 - 2x_2 + 2x_3 + 3x_4 - 5x_5 &= 0 \\ -2x_1 + 2x_2 - 2x_3 - 4x_4 + 8x_5 &= 0 \\ 5x_1 - 6x_2 + 9x_3 + 7x_4 - 14x_5 &= 0 \end{aligned}$$
   over $\mathbb{R}$. Indicate clearly the elementary operations performed.

4. Find all solutions of the system
   $$\begin{aligned} x_1 + x_2 + x_3 + x_7 + x_8 &= 1 \\ x_1 + x_2 + x_5 + x_7 + x_9 &= 1 \\ x_1 + x_2 + x_6 + x_7 + x_8 &= 1 \\ x_1 + x_3 + x_4 + x_5 + x_6 &= 1 \end{aligned}$$
   over the 2-element field $\mathbb{F}_2$. How many solutions does the system have? Justify all your assertions.