McGill University
Math 240: Discrete Structures 1
Solutions to Assignment 6

1. This is the same as the number of solutions of $x_1+x_2+x_3+x_4 = 11$ with $x_1 \leq 3$, $x_2 \leq 4$, $x_3 \leq 3$. If $N(x_i \geq a)$ is the number of solutions of $x_1 + x_2 + x_3 + x_4 = 11$ with $x_i \geq a$ the required number of solutions is, by inclusion-exclusion,

$$N(\text{all } x_i \geq 0) - N(x_1 \geq 4) - N(x_2 \geq 5) - N(x_3 \geq 4) \\ + N(x_1 \geq 4, x_2 \geq 5) + N(x_1 \geq 4, x_3 \geq 4) + N(x_2 \geq 5, x_3 \geq 4) - N(x_1 \geq 4, x_2 \geq 5, x_3 \geq 4)$$

which equals $C(14,11) - C(10,7) - C(9,6) - C(10,7) + C(5,2) + C(6,3) + C(5,2) = 80$.

2. We start with the edges of the complete graph K_4 on 4 vertices and remove edges one at a time, ignoring graphs isomorphic to previous ones, to get non-isomorphic graphs whose edge sets are given below.

   1. $E_1 = K_4 = \{12, 13, 14, 23, 24, 34\}$
   2. $E_2 = \{12, 23, 34, 41, 13\}$
   3. $E_3 = \{12, 23, 34, 41\}$, $E_4 = \{12, 23, 31, 41\}$
   4. $E_5 = \{12, 23, 34\}$, $E_6 = \{12, 31, 41\}$, $E_7 = \{12, 23, 31\}$
   5. $E_8 = \{12, 23\}$, $E_9 = \{12, 34\}$
   6. $E_{10} = \{12\}$
   7. $E_{10} = \{\}$

   That the graphs with the same number of edges are non-isomorphic is seen by computing the degrees of the vertices.

3. The edge sets of the given graphs G_1, G_2, G_3 are

$$E(G_1) = \{12, 13, 16, 25, 34, 37, 48, 56, 67, 78\},$$
$$E(G_2) = \{12, 14, 17, 23, 28, 36, 45, 57, 68, 78\},$$
$$E(G_3) = \{12, 14, 17, 23, 28, 36, 45, 56, 57, 68\},$$

   where ij denotes $\{i, j\}$. The mapping $f : V(G_1) \to V(G_2)$ given by

$$f(1) = 1, f(2) = 4, f(3) = 2, f(4) = 3, f(5) = 5, f(6) = 7, f(7) = 8, f(8) = 6$$

   gives a bijective map g:$E(G_1) \to E(G_2)$ where $g(ij) = g(i)f(j)$. Hence G_1 and G_2 are isomorphic. In G_3 each vertex of degree 3 is connected to one vertex of degree 3 while in G_1 or G_2 each vertex of degree 3 is connected to 2 others. Hence G_3 is not isomorphic to G_1 or G_2.

4. The required spanning subtrees are

$$G_1 - \{25, 73, 78\}, \quad G_2 - \{23, 45, 82\}, \quad G_3 - \{12, 14, 28\}.$$