McGill University
Math 240: Discrete Structures 1
Solutions to Assignment 5

1. (a) This is the same as the number of solutions to $x_1 + x_2 + x_3 + x_4 = 17 - 4 = 13$ with all $x_i \geq 0$ which equals $C(13 + 4 - 1, 13) = C(16, 13) = 560$.
   (b) This is the number of solutions to $x_1 + x_2 + x_3 + x_4 = 17$ with all $x_i \geq 0$ minus the number of those solutions with $x_1 \geq 6$. This equals $C(17 + 4 - 1, 17) - C(11 + 4 - 1, 11) = C(20, 17) - C(14, 11) = 776$.
   (c) This is the number of solutions of $x_1 + x_2 + x_3 + x_4 = 17$ with $x_1 \geq 0, x_2 \geq 9$ minus the number of solutions with $x_1 \geq 8, x_2 \geq 9$ which equals $C(8 + 4 - 1, 8) - C(0 + 4 - 1, 0) = C(11, 8) - C(3, 0) = 164$.
   (d) Since $1 + 2 + 4 + 4 = 11 < 17$ there are no solutions. Hence the number of solutions is 0.

2. The characteristic polynomial is $r^3 - 6r^2 + 11r - 6 = (r - 1)(r - 2)(r - 3)$. Since the roots are $1, 2, 3$ we have $a_n = A + B2^n + C3^n$. Using $a_0 = a_1 = a_2 = 1$, we find $A = 1, B = C = 0$ so that $a_n = 1$ for all n.

3. The associated homogeneous equation $a_{n+1} = a_{n+1} + 6a_n$ has characteristic equation $r^2 - r - 6 = (r+2)(r-3) =$ which gives $r = -2, 3$. Thus the associated homogenous equation has the general solution $x_n = A(-2)^n + B3^n$.

   The sequence whose n-th term is $b_n = 1 + n + 2^n$ satisfies a linear recurrence equation whose characteristic polynomial is $(r - 1)^2(r - 2)$. Any solution of the original equation is therefore a solution of an equation with characteristic polynomial $(r - 1)^2(r - 2)(r + 2)(r - 3)$ and hence is of the form

$$a_n = A(-2)^n + B3^n + C + Dn + E2^n.$$

   Substituting this in the original equation we get $C = -7/36, D = -1/6, E = -1/4$. Using the initial conditions $a_0 = 1, a_1 = 2$, we get $A = 53/180, B = 23/20$.

4. The required number is the number of surjective mappings (ball $i \mapsto$ urn j) of a set of 8 elements into a set with 3 elements. The number of such mappings is

$$3^8 - \binom{3}{0}(3 - 1)^8 + \binom{3}{2}(3 - 2)^8 = 5796.$$

5. The number of 7-letter words is $\frac{7!}{3!3!1!} = 140$. The number of 6-letter words using no R is $\frac{6!}{3!3!} = 20$. The number of 6-letter words using one less R or S is $\frac{6!}{3!2!} + \frac{6!}{3!2!} = 120$. The number of 5-letter words using no R and one less S or E is $\frac{5!}{3!2!} + \frac{5!}{3!2!} = 20$. The number of 5-letter words using one R and either one S or one E is $\frac{5!}{3!1!1!} + \frac{5!}{3!1!1!} = 40$. The number of 5-letter words using one R and one less S and one less E is $\frac{5!}{2!2!1!} = 30$. The total number of words is $140 + 20 + 120 + 20 + 40 + 30 = 370$.