McGill University
Math 240: Discrete Structures 1
Solutions to Assignment 3

1. (a) Let S be the set of natural numbers which are either even or odd. Then $0 \in S$ since $0 = 2 \cdot 0$. If $n \in S$ and $n = 2m$ then $n+1 = 2m+1 \in S$; if $n = 2m+1$ then $n+1 = 2m+2 = 2(m+1) \in S$. Hence S is an inductive set which implies $S = \mathbb{N}$.

   (b) If $5n+2$ is even and $n = 2k+1$ then $5n+2 = 10k+7 = 2(5k+3)+1$ which contradicts the fact that $5n+1$ is even.

2. We first prove that $f \circ f^n = f^n \circ f$ for all n by induction on n. For $n = 0$ we have

$$f \circ f^0 = f \circ 1_X = f = 1_X \circ f = f^0 \circ f$$

and if we assume that $f \circ f^n = f^n \circ f$ for some n then

$$f \circ f^{n+1} = f \circ (f \circ f^n) = f \circ (f^n \circ f) = (f \circ f^n) \circ f = f^{n+1} \circ f$$

using the fact that composition of functions is associative.

   (a) Proof by induction on n. For $n = 0$ we have

$$f^{m+0} = f^m = f^m \circ 1_X = f^m \circ f^0$$

   and assuming $f^{m+n} = f^m \circ f^n$ true for some n we have

$$f^{m+(n+1)} = f^{(m+n)+1} = f \circ f^{m+n} = f \circ (f^m \circ f^n) = (f \circ f^m) \circ f^n = (f^m \circ f) \circ f^n = f^m \circ (f \circ f^n) = f^m \circ f^{n+1}.$$

   (b) Proof by induction on n. For $n = 0$ we have

$$(f^m)^0 = 1_X = f^0 = f^{m \cdot 0}$$

   and assuming $(f^m)^n = f^{mn}$ true for some n we have

$$(f^m)^{n+1} = f^m \circ (f^m)^n = f^m \circ f^{mn} = f^{m+mn} = f^{m(n+1)}.$$

3. For $n =$ both sides are 0. Assuming the formula true for some n we have

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3 = n^2(n+1)^2/4 + (n+1)^3 = (n+1)^2(n^2+4n+4)/4 = (n+1)^2(n+2)^2/4.$$

4. Assume the statement is false and let n be the smallest n for which it is false. Then $n \geq 2$ and

$$\begin{aligned} a_n &= 5a_{n-1} - 6a_{n-2} = 5(2^{n-1} + 3^{n-1}) - 6(2^{n-2} + 3^{n-2}) \\ &= (5-3)2^{n-1} + (5-2)3^{n-1} = 2^n + 3^n \end{aligned}$$

which contradicts the fact that the statement is false for n.