McGill University
Math 240: Discrete Structures 1
Solutions to Assignment 2

1. (a) The relation $\{(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3)\}$ on $A = \{1,2,3\}$ is reflexive and symmetric but not transitive.

   (b) The relation $R = \{(1,1),(1,2),(2,2)\}$ on $A = \{1,2\}$ is reflexive and transitive but not symmetric.

   (c) The relation $R = \emptyset$ on $A = \{0\}$ is symmetric and transitive but not reflexive.

2. (a)

$$\begin{aligned} y \in f(A) \cap f(B) &\implies (\exists a, b) a \in A \wedge b \in B \wedge y = f(a) = f(b) \\ &\implies (\exists x \in A \cap B) y = f(x) \quad \text{(since } a = b \text{ by injectivity of } f\text{)} \\ &\implies y \in f(A \cap B) \end{aligned}$$

$$y \in f(A \cap B) \implies (\exists x \in A \cap B) y = f(x) \implies y \in f(A) \cap f(B)$$

$$\begin{aligned} y \in f(A - B) &\implies (\exists x) x \in A \wedge x \notin B \wedge y = f(x) \\ &\implies y \in f(A) - f(B) \quad \text{(since } y = f(b) \text{ with } b \in B \implies x = b \in B \text{ by injectivity of } f\text{)} \end{aligned}$$

$$y \in f(A) - f(B) \implies (\exists x \in A) y = f(x) \wedge y \notin f(B) \implies (\exists x \in A - B) y = f(x) \implies y \in f(A - B)$$

   (b) $x \in f^{-1}(A \cap B) \iff f(x) \in A \cap B \iff f(x) \in A \wedge f(x) \in B \iff x \in f^{-1}(A) \wedge f^{-1}(B)$
   $x \in f^{-1}(A - B) \iff f(x) \in A - B \iff f(x) \in A \wedge f(x) \notin B \iff x \in f^{-1}(A) - f^{-1}(B)$.

3. Define $F : P(X) \to 2^X$ by $F(A) = \chi_A$. Then F is injective since $F(A) = F(B) \implies \chi_A = \chi_B$ which implies $A = B$ since $x \in A \iff \chi_A(x) = 1 \iff \chi_B(x) = 1 \iff x \in B$. To prove surjectivity of F let $f \in 2^X$ and let $A = f^{-1}(1)$. Then $f = F(A) = \chi_A$ since $x \in A \implies f(x) = 1 = \chi_A(x)$ and $x \notin A \implies f(x) = 0 = \chi_A(x)$.

4. (a) $f(a) = f(b) \implies g(f(a)) = g(f(b)) \implies g \circ f(a) = g \circ f(b) \implies a = b$ since $g \circ f$ is injective.

   (b) Let $z \in Z$. Then $g \circ f$ surjective implies $z = g \circ f(x) = g(f(x))$ for some $x \in X$. Setting $y = f(x)$, we have $y \in Y$ and $g(y) = z$.

   (c) $g \circ f = 1_X$ is bijective and so f is injective, g is surjective. $f \circ g = 1_Y$ is bijective and so f is surjective, g injective. Hence f, g are bijective. Since $y = f(x) \iff g(y) = x$ we see that $f = g^{-1}$ and $g = f^{-1}$.