McGill University
Math 240: Discrete Structures 1
Solutions to Assignment 1

1. (a)

p	q	r	$p \vee q$	$\neg p$	$\neg p \vee r$	$(p \vee q) \wedge (\neg p \vee r)$	$q \vee r$	$(p \vee q) \wedge (\neg p \vee r) \rightarrow q \vee r$
T	T	T	T	F	T	T	T	T
T	T	F	T	F	F	F	T	T
T	F	T	T	F	T	T	T	T
T	F	F	T	F	F	F	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T	T
F	F	T	F	T	T	F	T	T
F	F	F	F	T	T	F	T	T

(b)

p	q	r	$p \vee q$	$p \rightarrow r$	$q \rightarrow r$	$(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)$	$(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r) \rightarrow r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

2. It suffices to prove (a) $\iff$ (b) and (c) $\iff$ (a).

$$\begin{aligned}
p \leftrightarrow q &\iff (p \rightarrow q) \wedge (q \rightarrow p) \quad \text{(Definition of } p \leftrightarrow q\text{)}\\
&\iff (q \vee \neg p) \wedge (p \vee \neg q) \quad \text{(Definition of } p \rightarrow q \text{ and } q \rightarrow p\text{)}\\
&\iff ((q \vee \neg p) \wedge p) \wedge ((q \vee \neg p) \wedge \neg q) \quad \text{(Distributive Law)}\\
&\iff ((q \wedge p) \vee (\neg p \wedge p)) \vee ((q \wedge \neg q) \vee (\neg p \wedge \neg q) \quad \text{(Distributive Law)}\\
&\iff (p \wedge q) \vee (\neg p \wedge \neg q) \quad \text{(Commutative, Negation and Identity Laws)}
\end{aligned}$$

$$\begin{aligned}
\neg(p \oplus q) &\iff \neg((p \wedge \neg q) \vee (q \wedge \neg p)) \quad \text{(Definition of } p \oplus q\text{)}\\
&\iff (\neg(p \wedge \neg q) \wedge \neg(q \wedge \neg p) \quad \text{(DeMorgan)}\\
&\iff (\neg p \vee q) \wedge (p \vee \neg q) \quad \text{(DeMorgan and Double Negation)}\\
&\iff p \leftrightarrow q \quad \text{(Definition of } p \leftrightarrow q\text{)}
\end{aligned}$$

3. (a) $\forall(A, B, C)((A \cup C = B \cup C) \rightarrow (A = B))$ is false. Counterexample: $A = \emptyset, B = C = \{0\}$.
   (b) $\forall(A, B, C)((A \cap C = B \cap C) \rightarrow (A = B))$ is false. Counterexample: $A = C = \emptyset, B = \{0\}$.

(c) $\forall(A,B,C)((A\oplus B)\cap C=(A\cap C)\oplus(B\cap C))$ is true. If U is any set containing A,B,C and $\overline{X}=U-X$, we have $X\oplus Y=(X\cap\overline{Y})\cup(Y\cap\overline{X})=Y\oplus X$ and

$$\begin{aligned}
(A\cap C)\oplus(B\cap C) &= ((A\cap C)\cap\overline{B\cap C})\cup((B\cap C)\cap\overline{A\cap C}) \quad \text{(Definition of } X\oplus Y\text{)}\\
&= ((A\cap C)\cap(\overline{B}\cup\overline{C}))\cup((B\cap C)\cap(\overline{A}\cup\overline{C})) \quad \text{(DeMorgan)}\\
&= ((A\cap C\cap\overline{B})\cap(A\cap C\cap\overline{C}))\cup((B\cap C\cap\overline{A})\cap(B\cap C\cap\overline{C})) \quad \text{(Distributive Law)}\\
&= (A\cap C\cap\overline{B})\cup(B\cap C\cap\overline{A})\\
&= ((A\cap\overline{B})\cup(B\cap\overline{A})\cap C \quad \text{(Distributive Law)}\\
&= (A\oplus B)\cap C
\end{aligned}$$

(d) $\forall(A,B,C)((A\oplus B)\cup C=(A\cup C)\oplus(B\cup C))$ is false. Counterexample: $A=B=\emptyset, C=\{0\}$.

4. Let $U=A\cup B\cup C$ and let $\overline{X}=U-X$. Then

$$\begin{aligned}
A\oplus(B\oplus C) &= (A\cap\overline{(B\cap\overline{C})\cup(C\cap\overline{B})})\cup(((B\cap\overline{C}))\cup(C\cap\overline{B}))\cap\overline{A})\\
&= (A\cap((\overline{B}\cup C)\cap(\overline{C}\cup B))\cup(((B\cap\overline{C}))\cup(C\cap\overline{B}))\cap\overline{A}) \quad \text{(DeMorgan)}\\
&= (A\cap B\cap C)\cup(A\cap\overline{B}\cap\overline{C})\cup(B\cap\overline{C}\cap\overline{A})\cup(C\cap\overline{A}\cap\overline{B}) \quad \text{(Distributive Law)}\\
&= C\oplus(A\oplus B) \quad \text{(Symmetry in } A,B,C\text{)}\\
&= (A\oplus B)\oplus C
\end{aligned}$$