McGill University
Math 240: Discrete Structures 1
Assignment 3: due Friday, October 14, 2005

Reading: Text 1.5: Methods of Proof, 3.2: Sequences and Summation, 3.3: Mathematical Induction. 3.2(first 3 pages) and set theory notes: Recursive Definitions

Questions:

1. A natural number is said to be even if it is of the form $2n$ for some natural number n; it is said to be odd if it is of the form $2n+1$ for some natural number n.
   (a) Prove by induction that every natural number is either even or odd.
   (b) Prove by contradiction that if n is a natural number with $5n+2$ even then n is even.

2. If $f : X \to X$ is a mapping then for any natural number n the mapping $f^n : X \to X$ is defined inductively by $f^0 = 1_X$ and $f^{n+1} = f \circ f^n$. Prove the following by induction on n.
   (a) $(\forall m, n \in \mathbb{N})\ f^{m+n} = f^m \circ f^n$;
   (b) $(\forall m, n \in \mathbb{N})\ (f^m)^n = f^{mn}$.

3. Prove by induction on n that $\sum_{k=1}^{n} k^3 = n^2(n+1)^2/4$ for any natural number n.

4. The sequence $(a_n)_{n\in\mathbb{N}}$ is defined inductively by

$$a_0 = 2, a_1 = 5, a_{n+2} = 5a_{n+1} - 6a_n \text{ for } n \geq 0.$$

   Prove by induction that $a_n = 2^n + 3^n$ for all n.