McGill University
MATH 236: Midterm Test Solutions

1. (a) The zero vector of V is the zero function which belongs to S since $f(x) = 0$ for all x implies $f(2-x) = xf(x) = 0$ for all x. If $f, g \in S$ we have $(af+bg)(2-x) = af(2-x)+bg(2-x) = axf(x)+bxg(x) = x(af+bg)(x)$ which implies that $af + bg \in S$. Hence S is a subspace.

   (b) S is not a subspace since $(1,1), (1,-1) \in S$ but $(1,1)+(1,-1) = (2,0) \notin S$.

2. (a) Since $(x-1)^2$, $(x+1)^2$, $(x-2)^2$, $(x+2)^2$ is a sequence of length 4 in $\mathrm{Span}(1, \mathrm{x}, \mathrm{x}^2)$, a 3-dimensional vector space, the given sequence must be linearly dependent.

   (b) Since $ae^x + be^{2x} + ce^{3x} = 0$ in $\mathrm{Ker}(D-1) + \mathrm{Ker}(D-2) + \mathrm{Ker}(D-3)$ with $e^{kx} \in \mathrm{Ker}(D-k)$ we have $ae^x = be^{2x} = ce^{3x} = 0$ since the sum is direct. Setting $x = 0$, we get $a = b = c = 0$ so that the given sequence is linearly independent.

3. (a) Let V be a vector space. The kernel of a linear mapping with domain V is the set of vectors v in V with $T(v) = 0$. It is a subspace of V. A scalar λ is an eigenvalue of a linear operator T on V if there is a non-zero vector $v \in V$ with $T(v) = \lambda v$.

   (b) If $T(v) = \lambda v$ then $T^2(v) = T(\lambda v) = \lambda T(v) = \lambda^2 v$ so that $(T^2+aT+b)(v) = T^2(v)+aT(v)+bv = \lambda^2 v+a\lambda v+bv = (\lambda^2+a\lambda+b)v$. Hence $T^2+aT+b = 0$ implies $(\lambda^2 + a\lambda + b)v = 0$ which in turn implies $\lambda^2 + a\lambda + b = 0$ if $v \neq 0$. Hence, if λ is an eigenvalue of T and $T^2+aT+b = 0$, we have $\lambda^2+a\lambda+b = 0$.

4. (a) Since $T((x_1, y_1, z_1)+(x_2, y_2, z_2)) = T(x_1+x_2, y_1+y_2, z_1+z_2) = (2(x_1+x_2)+y_1+y_2+z_1+z_2, x_1+x_2+2(y_1+y_2)+z_1+z_2, x_1+x_2+y_1+y_2+2(z_1+z_2)) = (2x_1+y_1+z_1, x_1+2y_1+z_1, x_1+y_1+2z_1)+(2x_2+y_2+z_2, x_2+2y_2+z_2, x_2+y_2+2z_2) = T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$ and $T(a(x,y,z)) = T(ax, ay, az) = (2ax + ay + az, ax + 2ay + az, ax + ay + 2az) = aT(x,y,z)$ we see that T is linear. Since $T^2(x,y,z) = T(2x + y + z, x + 2y + z, x + y + 2z) = (6x + 5y + 5y, 5x + 6y + 5z, 5x + 5y + 6z)$ we have $(T^2 - 5T + 4)(x,y,z) = (6x + 5y + 5z, 5x + 6y + 5z, 5x + 5y + 6z) - 5(2x + y + z, x + 2y + z, x + y + 2z) + 4(x,y,z) = (0,0,0)$.

   (b) Since $T^2 - 5T + 4 = 0$ the possible eigenvalues of T are $1, 4$, the roots of $x^2 - 5x + 4 = 0$. Now $(T-1)(x,y,z) = (x+y+z, x+y+z, x+y+z)$ implies that $\mathrm{Ker}(T-1) = \mathrm{Span}((1,-1,0),(1,0,-1)$. Since $T^2 - 5T + 4 = (T-1)(T-4) = 0$ we see that $\mathbb{R}^3 = \mathrm{Ker}((T-1)(T-4) = \mathrm{Ker}(T-1) \oplus \mathrm{Ker}(T-5)$ we see that $\dim \mathrm{Ker}(T-4) = 1$. Since $T(1,1,1) = 4(1,1,1)$ we see that $\mathrm{Ker}(T-4) = \mathrm{Span}((1,1,1))$. Since $(1,-1,0),(1,0,-1)$ is a basis for $\mathrm{Ker}(T-1)$ and $(1,1,1)$ is a basis for $\mathrm{Ker}(T-4)$ we see that $(1,-1,0),(1,0,-1),(1,1,1)$ is a basis for $\mathbb{R}^3$ consisting of eigenvectors of T.