MATH 236: Solutions for Assignment 5

1. (a) If $A=[a_{ij}], B=[b_{ij}]$ we have $T(A+B)=\sum_{i=1}^n(a_{ii}+b_{ii})=\sum_{i=1}^n a_{ii}+\sum_{i=1}^n b_{ii}=\mathrm{tr}(A)+\mathrm{tr}(B)$ and $\mathrm{tr}(cA)=\sum_{i=1}^n c\,a_{ii}=c\sum_{i=1}^n a_{ii}=c\,\mathrm{tr}(A)$ for any scalar c. Also $\mathrm{tr}(AB)=\sum_{i,j=1}^n a_{ij}b_{ji}=\mathrm{tr}(BA)$.

   (b) If E_{ij} is the $n\times n$ matrix with (i,j)-th entry 1 and all other entries equal to 0, we have $E_{ij}E_{jk}=E_{ik}$ and $E_{ij}E_{km}=0$ if $j\neq k$. The kernel of the trace map has dimension n^2-1 with basis the matrices E_{ij} $(i\neq j)$, $E_{11}-E_{ii}$ $(i\neq 1)$. If $i\neq j$ we have $E_{ij}=E_{ii}E_{ij}-E_{ij}E_{ii}$ and we have $E_{11}-E_{ii}=E_{1i}E_{i1}-E_{i1}E_{1i}$ which shows that Ker(tr) is spanned by matrices of the form $AB-BA$. This shows that $\mathrm{tr}(A)=0$ implies $\phi(A)=0$. If A is any $n\times n$ matrix we have $A=\mathrm{tr}(A)E_{11}+C$ with $\mathrm{tr}(C)=0$ which implies $\phi(A)=c\,\mathrm{tr}(A)$ with $c=\phi(E_{11})$ and hence $\phi=c\,\mathrm{tr}$.

2. (a) For fixed (y_1,y_2) we have that $\langle(x_1,x_2),(y_1,y_2)\rangle=x_1y_1+2x_1y_2+2x_2y_1+5x_2y_2$ is linear in (x_1,x_2) and $\langle(x_1,x_2),(y_1,y_2)\rangle=\langle(y_1,y_2),(x_1,x_2)\rangle$. Since $\langle(x_1,x_2),(x_1,x_2)\rangle=(x_1+2x_2)^2+x_2^2$ we see that $\langle(x_1,x_2),(x_1,x_2)\rangle\geq 0$ with equality if and only if $(x_1,x_2)=(0,0)$.

   (b) For fixed (y_1,y_2) we have that $\langle(x_1,x_2),(y_1,y_2)\rangle=x_1\bar y_1+ix_1\bar y_2-ix_2\bar y_1+2x_2\bar y_2$ is linear in (x_1,x_2) and $\langle(x_1,x_2),(y_1,y_2)\rangle=\overline{\langle(y_1,y_2),(x_1,x_2)\rangle}$. Since $\langle(x_1,x_2),(x_1,x_2)\rangle=|x_1-ix_2|^2+|x_2|^2$ we see that $\langle(x_1,x_2),(x_1,x_2)\rangle\geq 0$ with equality if and only if $(x_1,x_2)=(0,0)$.

3. (a) An orthonormal basis for $W=\mathrm{Span}(1,x,x^2)$ is $f_1(x)=1, f_2(x)=x-\frac12, f_3(x)=x^2-x+\frac16$. We have $||f_1||=1$, $||f_2||=1/2\sqrt{3}$, $||f_3||=1/6\sqrt{5}$. The best approximation to $h(x)=\sin(\pi x)$ by a function in W is the orthogonal projection of h on W. This is the function
$h(x)=\frac{\langle h,f_1\rangle}{\langle f_1,f_1\rangle}f_1+\frac{\langle h,f_2\rangle}{\langle f_2,f_2\rangle}f_2+\frac{\langle h,f_3\rangle}{\langle f_3,f_3\rangle}f_3=\frac{2}{\pi}+\frac{60(\pi^2-12)}{\pi^3}(x^2-x+\frac16)$.

   (b) If $g=af_1+bf_2+cf_3$ then $f(1)=\langle f,g\rangle$ for all $f\in W$ gives $1=f_1(1)=a$, $1/2=b/12$, $1/6=c/180$ so that $a=1$, $b=6$, $c=30$. Hence $g(x)=30x^2-24x+3$.

4. (a) First note that $A=\begin{bmatrix}1&i\\-i&1\end{bmatrix}$ is self-adjoint. Then $\langle X,T(Y)\rangle=\mathrm{tr}(X(AY-YA)^*)=$
$\mathrm{tr}(X(AY^*-Y^*A))=\mathrm{tr}(XAY^*)-\mathrm{tr}(XY^*A)=\mathrm{tr}(XAY^*)-\mathrm{tr}(AXY^*)=\mathrm{tr}((AX-XA)Y^*)=\langle T(X),Y\rangle$.

   (b) If $X=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ we have $T(X)=i\begin{bmatrix}b+c&d-a\\d-a&-b-c\end{bmatrix}=i(b+c)\begin{bmatrix}1&0\\0&-1\end{bmatrix}+i(d-a)\begin{bmatrix}0&1\\1&0\end{bmatrix}$ which implies $\mathrm{Im}(T)=\mathrm{Span}(E=\begin{bmatrix}1&0\\0&-1\end{bmatrix},F=\begin{bmatrix}0&1\\1&0\end{bmatrix})$ and $\mathrm{Ker}(T)=\mathrm{Span}(\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\-1&0\end{bmatrix})$. The subspace $W=\mathrm{Im}(T)$ is T-invariant and $T(E)=-2iF$, $T(F)=2iE$ implies that the matrix of the restriction of T to W with respect to the basis E,F is the matrix $B=\begin{bmatrix}0&2i\\-2i&0\end{bmatrix}$. The minimal polynomial of B is x^2- so that $2,-2$ are the eigenvalues of B. Eigenvectors for these eigenvalues are $\begin{bmatrix}i\\1\end{bmatrix}$ and $\begin{bmatrix}1\\i\end{bmatrix}$ respectively. Thus the matrices $iE+F$ and $E+iF$ are eigenvectors of T with eigenvalues $2,-2$ respectively. It follows that the required orthonormal basis of eigenvectors of T is
$$F_1=\frac12\begin{bmatrix}i&1\\1&-i\end{bmatrix},\ F_2=\frac12\begin{bmatrix}1&i\\i&-1\end{bmatrix},\ F_3=\frac{1}{\sqrt2}\begin{bmatrix}1&0\\0&1\end{bmatrix},\ F_4=\frac{1}{\sqrt2}\begin{bmatrix}0&1\\-1&0\end{bmatrix}.$$

   (c) The matrix of T with respect to the basis $E_1=\begin{bmatrix}1&0\\0&0\end{bmatrix},E_2=\begin{bmatrix}0&1\\0&0\end{bmatrix},E_3=\begin{bmatrix}0&0\\1&0\end{bmatrix},F_4=\begin{bmatrix}0&0\\0&1\end{bmatrix}$ is the matrix $A=\begin{bmatrix}0&i&i&0\\-i&0&0&i\\-i&0&0&i\\0&-i&-i&0\end{bmatrix}$. The unitary matrix $U=\begin{bmatrix}i/2&1/2&1/\sqrt2&0\\1/2&i/2&0&1/\sqrt2\\1/2&i/2&0&-1/\sqrt2\\-i/2&-1/2&1/\sqrt2&0\end{bmatrix}$ whose columns are the coordinate vectors of F_1,F_2,F_3,F_4 satisfies $U^{-1}AU=$diag$(2,-2,0,0)$, the diagonal matrix with entries $2,-2,0,0$, since $T(F_1)=2F_2$, $T(F_2)=-2F_2$, $T(F_3)=T(F_4)=0$.