Math 236: Algebra 2
Solutions to Assignment 4

1. (a) $A^2-(a+d)A+(ad-bc)I = \begin{bmatrix} a^2+bc-(a+d)a+ad-bc & ab+bd-(a+d)b \\ ac+cd-(a+d)c & bc+d^2-(a+d)d+ad-bc \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

   (b) If $m(x) = x^2-(a+d)x+(ad-bc)$ we have $m(A) = 0$ by (a) and so the minimal polynomial $m_A(x)$ of A divides $m(x)$. Since A is not a scalar multiple of the identity matrix the degree of $m_A(x)$ is not 1 and so $m_A(x) = m(x)$ since both polynomials are monic and of the same degree.

   (c) If $m_A(x) = (x-\lambda)^2$ we have $A-\lambda I \neq 0$ which implies the existence of a column matrix P_2 with $P_1 = (A-\lambda I)P_2 \neq 0$.Then $AP_1 = \lambda P_1$, $AP_2 = P_1+\lambda P_1$. If P is the matrix whose columns are P_1, P_2 then P is invertible since P_2 is not a scalar multiple of P_1 and $P^{-1}AP = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = B$ since B is the matrix of the linear operator T_A on $\mathbb{F}^{2\times 1}$ defined by $T_A(X) = AX$ with respect to the basis P_1, P_2. Conversely if $P^{-1}AP = B$ then B is the matrix of T_A with respect to the basis formed by the columns of P so that the minimal polynomial of T_A is equal to $m_B(x) = (x-\lambda)^2$.

2. We are given that $x_{n+1} = x_n/2+y_n/3$, $y_{n+1} = x_n/2+2y_n/3$ which proves the assertion. It follows by induction that

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix}^{n-1} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

The minimal polynomial of $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix}$ is $x^2-(7/6)x+1/6 = (x-1)(x-1/6)$ so that the eigenvalues of A are 1 and 1/6. Since $\mathrm{Null(A-1)} = \mathrm{Span}(\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathrm{Null(A-(1/6)I)} = \mathrm{Span}(\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ we see that $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 1/6 \end{bmatrix}$ where $P = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$. Hence $A = P\begin{bmatrix} 1 & 0 \\ 0 & 1/6 \end{bmatrix}P^{-1}$ so that $A^n = P\begin{bmatrix} 1 & 0 \\ 0 & 1/6 \end{bmatrix}^n P^{-1} = P\begin{bmatrix} 1 & 0 \\ 0 & 1/6^n \end{bmatrix}P^{-1}$. Hence $\lim_{n\to\infty} A^n = P\begin{bmatrix} 1 & 0 \\ 0 & 1/6 \end{bmatrix}^n P^{-1} = P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}P^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{3}{5} \end{bmatrix}$ so that $\lim_{n\to\infty} \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$.

3. Since $m_A(x) = x^2-(a+d)x+ad-bc$ we have $m_A(1) = 1-(a+d)+ad-bc = 1-(a+1-b)+a(1-b)-b(1-a) = 0$. Hence $m_A(x) = (x-1)(x-r)$. Since $a+d = 1+r$ and $0 < a+d < 2$ we have $-1 < r < 1$. Since $m_A(x)$ is a product of distinct linear factors there is an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix with diagonal entries $1, r$. It follows as above that $B = \lim_{n\to\infty} A^n$ exists. Since $AB = B$ the columns of B are eigenvectors of A with eigenvalue 1. Also, since the product of stochastic matrices is stochastic, it follows that B is stochastic. Hence, if $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is the first column of B it is an eigenvector of A with $X = \alpha+\beta = 1$. The second column of B must be cX for some scalar c since $\dim \mathrm{Null(A-I)} = 1$. Now $c\alpha + c\beta = 1$ implies $c = 1$.

4. (a) We have $(1+T+T^2+\cdots+T^{n-1})(1-T) = (1-T)(1+T+T^2+\cdots+T^{n-1}) =$
   $1+T+T^2+\cdots+T^{n-1}-(T+T^2+\cdots+T^n) = T^n = 0$.

   (b) Since $(T-b) = (T-a+a-b) = (a-b)(1-S)$ with $S = (T-a)/(b-a)$. Since $S^n = 0$ we see by (a) that $1-S$ is invertible and hence $T-b$ is invertible with $(T-b)^{-1} = (a-b)^{-1}(1+S+\ldots+S^{n-1})$.

   (c) Since $W = \mathrm{Null(D-2)^2} = \mathrm{Span(e^{2x}xe^{2x})}$ is D-invariant we can restrict D to W to get a linear operator R on W with $(R-2)^2 = 0$. By (b) the operator $R-1$ is invertible with $(R-1)^{-1} = 1-(R-2)$. Hence $(R-1)^2$ is invertible with $(R-1)^2 = (1-(R-2))^2 = 1-2(R-2) = 5-2R$. Hence $y_P = (3-2R)(xe^2) = 5xe^{2x}-2D(xe^{2x}) = xe^{2x}-2e^{2x}$ is a solution of the given differential equation.

   (d) Since $(D-2)^2(y-y_P) = (D-2)^2y-(D-2)^2y_P = 0$ we have $y-y_P \in \mathrm{Span(e^{2x}, xe^{2x})}$ so that $y = y_P + Ae^{2x} + Bxe^{2x}$.

5. We have

$$W_1 = \mathrm{Null(A-I)} = \mathrm{Span}\left(\mathrm{X_1} = \begin{bmatrix}1\\1\\-1\\0\end{bmatrix}, \mathrm{X_2} = \begin{bmatrix}1\\1\\0\\-1\end{bmatrix}\right), \quad \mathrm{W_2} = \mathrm{Null(A+I)} = \mathrm{Span}\left(\mathrm{X_3} = \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \mathrm{X_4} = \begin{bmatrix}0\\1\\0\\-1\end{bmatrix}\right)$$

and $BX_1 = X_1 + 4X_2, BX_2 = 2X_2 + 3X_2, BX_3 = 2X_3 + X_4, BX_4 = X_3 + 2X_4$. If $T_(X) = BX$ and R_1, R_2 are the restrictions of T to W_1 and W_2 we have

$$C = [R_1]_{(X_1,X_2)} = \begin{bmatrix}1 & 2\\4 & 3\end{bmatrix}, \quad D = [R_2]_{(X_3,X_4)} = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}.$$

Now $m_C(x) = x^2 - 4x - 5 = (x-5)(x+1)$, $m_D(x) = x^2 - 2x - 3 = (x-3)(x+1)$ so that C and D are diagonalizable. Since

$$\mathrm{Null(C-5I)} = \mathrm{Span}(\begin{bmatrix}1\\2\end{bmatrix}),\ \mathrm{Null(C+I)} = \mathrm{Span}(\begin{bmatrix}1\\-1\end{bmatrix}),\ \mathrm{Null(D-3I)} = \mathrm{Span}(\begin{bmatrix}1\\1\end{bmatrix}),\ \mathrm{Null(C+I)} = \mathrm{Span}(\begin{bmatrix}1\\-1\end{bmatrix})$$

we see that

$$P_1 = x_1 + 2x_2 = \begin{bmatrix}3\\3\\-1\\-2\end{bmatrix},\ P_2 = X_1 - X_2 = \begin{bmatrix}0\\0\\-1\\1\end{bmatrix},\ P_3 = X_3 + X_4 = \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix},\ P_4 = X_3 - X_4 = \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$$

are eigenvectors of B with eigenvalues $5, -1, 3, -1$ respectively. Since they are also eigenvectors of A with eigenvalues $1, 1, -1, -1$ respectively, we see that the matrix P whose columns are P_1, P_3, P_3, P_4 is an invertible matrix which diagonalizes both A and B.

6. (a) If $u \in \mathrm{Ker(p(S))}$ then $p(S)(u) = 0$ which implies $p(S)T(u) = Tp(S)(u) = 0$ so that $T(u) \in \mathrm{Ker(p(S))}$. If $u \in \mathrm{Im(p(S))}$ then $u = p(S)(v)$ for some v and $T(u) = Tp(S)(v) = p(S)T(v) \in \mathrm{Im(p(S)}$.
   (b) Let R be the restriction of T to W. Since W is T-invariant, R is a linear operator on W. Since $m_T(R) = 0$ we have $m_R(x) \mid m_T(x)$. But T diagonalizable implies that $m_T(x)$ is a product of distinct linear factors and hence the same is true of $m_R(x)$. Thus R is diagonalizable.
   (c) Since S is diagonalizable V is the direct sum of the eigenspaces of S. Since T commutes with S each of the eigenspaces of S are T-invariant. Hence, using (b), we see that T is diagonalizable if and only if the restriction of T to each eigenspace of S is diagonalizable.

7. (a) Let $W_k = \mathrm{Ker((L-a)^k)} = \mathrm{Span((a^{n-1}), (na^{n-1}), \ldots, (n^{k-1}a^{n-1}))}$. Then $(L-a)(W_k) \subseteq W_{k-1}$ since

$$(L-1)((n^{i-1}a^{n-1})) = (((n+1)^{i-1}a^n - n^{i-1}a^n) = (\sum_{j<i} c_j n^j a_n) \in W_{i-1}$$

   for $i \geq 1$. It follows that $(L-a)^k(W_k) \subseteq W_0 = \{0\}$. Hence $W_k \subseteq \mathrm{Ker(L-a)^k}$ and we must have equality since $\dim \mathrm{Ker(L-a)^k} = \mathrm{k} = \dim \mathrm{W_k}$ since $(a^{n-1}), (na^{n-1}), \ldots, (n^{k-1}a^{n-1})$ is a linearly independent sequence. Indeed, $A_1 a^{n-1} + A_2 n a^{n-1} + \ldots + A_k n^{k-1} a^{n-1} = 0$ for all $n \geq 1$ if and only if $A_1 + A_2 n + \ldots + A_k n^{k-1} = 0$ for all $n \geq 1$ which implies $A_1 = \cdots = A_k = 0$ since otherwise $A_1 + A_2 x + \cdots + A_k x^{k-1}$ would be a polynomial of degree $\leq k-1$ with more than $k-1$ roots.
   (b) $(L-1)(s) = ((n+1)^2 3^{n+1}) = 9(n^2 3^{n-1} + 18(n3^{n-1}) + 9(3^{n-1}) \in \mathrm{Ker(L-3)^3} \implies$ $s \in \mathrm{Ker(L-1)^2(L-1))} = \mathrm{Span((1), (3^{n-1}), (n3^{n-1}), (n^2 3^{n-1}))} \implies \mathrm{s_n = A + B3^{n-1} + Cn3^{n-1} + Dn^2 3^{n-1}}$ for unique scalars A, B, C, D. Using the fact that $s_1 = 3, s_2 = 39, s_3 = 282, s_4 = 1578$ we get $A = -3/2, B = 9/2, C = -9/2, D = 9/2$.

8. We have $\frac{dX}{dt} = AX$ with $A = \begin{bmatrix}-2 & 1\\1 & -2\end{bmatrix}$ and $m_A(x) = x^2 + 4x + 3 = (x+1)(x+3)$, $\mathrm{Null(A+I)} = \begin{bmatrix}1\\-1\end{bmatrix}$, $\mathrm{Null(A+3I)} = \begin{bmatrix}1\\1\end{bmatrix}$. Setting $X = PY$ with $P = \begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}$ we get $\frac{dY}{dt} = P^{-1}APY$ from which $\frac{dy_1}{dt} = -y_1$, $\frac{dy_2}{dt} = -3y_2$ and so $y_1 = Ae^{-x}$, $y_2 = Be^{-3x}$. Hence $x_1 = y_1 + y_2 = Ae^{-x} + Be^{-3x}$, $x_2 = y_1 - y_2 = Ae^{-x} - Be^{-2x}$.