## McGill University Solution Sheet for MATH 236 Assignment 1

- 1. (a) We have to verify the vector space axioms for the operations  $\oplus, \odot$ :
  - (I) Associative Law:  $((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3) = (x_1 + x_2 1, y_1 + y_2 + 2) \oplus (x_1 + x_2 1, y_2 + 2) \oplus (x_2 + x_2 1, y_2 + 2) \oplus (x_3 + x_2 1) \oplus (x_3 + x_3 x_3 1) \oplus (x_3 + x_3 1) \oplus (x_3 + x_3 1)$
  - $(x_1 + x_2 + x_3 2, y_1 + y_2 + y_3 + 4) = (x_1, y_1) \oplus (x_2 + x_3 1, y_2 + y_3 + 2) = (x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3)).$
  - (II) Commutative Law:  $(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2 1, y_1 + y_2 + 2) = (x_2 + x_1 1, y_2 + y_1 + 2) = (x_2, y_2) \oplus (x_1, y_1).$
  - (III) Existence of Zero: The zero element is (1, -2) since  $(x, y) \oplus (1, -2) = (x + 1 1, y 2 + 2) = (x, y)$ .
  - (IV) Existence of Additive Inverses: The additive inverse of (x, y) is (-x + 2, -y 4) since
  - $(x,y) \oplus (-x+2, -y-4) = (x-x+2-1, y-y-4+2) = (1, -2).$
  - (V) Multiplication by 1:  $1 \odot (x, y) = (x + 1 1, y + 2 2) = (x, y).$
  - (VI) Associativity of Scalar Multiplication:  $a \odot (b \odot (x, y)) = a \odot (bx + 1 b, by + 2b 2) =$
  - $(abx + a ab + 1 a, aby + 2ab 2a + 2a 2) = (abx + 1 ab, aby + 2ab 2) = (ab) \odot (x, y).$

(VII) **Distributive Laws:** (a)  $(a + b)(x, y) = ((ax + bx + 1 - a - b, ay + by + 2a + 2b - 2) = (ax + 1 - a, ay + 2a - 2) \oplus (by + 1 - b, by + 2b - 2) = a \odot (x, y) \oplus b \odot (x, y);$ 

(b)  $a \odot ((x_1, y_1) \oplus (x_2, y_2)) = a \odot (x_1 + x_2 - 1, y_1 + y_1 + 2) = (ax_1 + ax_2 - a + 1 - a, ay_1 + ay_1 + 2a + 2a - 2) = (ax_1 + 1 - a, ay_1 + 2a - 2) \oplus (ax_2 + 1 - a, ay_2 + 2a - 2) = a \odot (x_1, y_1) \oplus a \odot (x_2, y_2).$ 

- (b) To show that T is an isomorphism of  $(\mathbb{R}^2, \oplus, \odot)$  to  $(\mathbb{R}^2, +, \cdot)$  we show (i) T is **injective:**  $T(x_1, y_1) = T(x_2, y_2) \implies (x_1 - 1, y_1 + 2) = (x_2 - 1, y_2 + 2) \implies x_1 - 1 = x_2 - 1, y_1 + 2 = y_2 + 2 \implies x_1 = x_2, y_2 = y_2 \implies (x_1, y_1) = (x_2, y_2).$ (ii) T is **surjective:** (x, y) = T(x - 1, y + 2)(iii) T is **linear:**  $T(a \odot (x_1, y_1) \oplus b \odot (x_2, y_2) = T(ax_1 + bx_2 + 1 - a - b, ay_1 + by_2 + 2a + 2b - 2) = (ax_1 + bx_2 - a - b, ay_1 + by_2 + 2a + 2b) = (ax_1 - a, ay_1 + 2a) + (bx_2 - b, by_2 + 2b) = a(x_1 - 1, y_1 + 2) + b(x_2 - 1, y_2 + 2) = aT(x_1, y_1) + bT(x_2, y_2).$
- 2. (a)  $0 \in W$  since  $x_n = 0$  for all  $n \ge 1$  implies  $x_{n+3} = x_{n+2} + nx_n = 0$  for all  $n \ge 1$ . If  $a, b \in \mathbb{F}$ ,  $x, y \in W$  then  $(ax + by)_{n+3} = ax_{n+3} + by_{n+3} = ax_{n+2} + nbx_n + ay_{n+2} + nby_n = (ax + by)_{n+2} + n(ax + by)_n$  which implies  $ax + by \in W$ . Hence W is a subspace of V.
  - (b)  $0 \in W$  since f(x) = 0 for all x implies  $f(x) = f(2x^2 + 1) = 0$  for all x. If  $a, b \in \mathbb{F}, f, g \in W$  then  $(af + bg)(x) = af(x) + bg(x) = af(2x^2 + 1) + bg(2x^2 + 1) = (af + bg)(2x^2 + 1)$  which implies  $af + bg \in W$ . Hence W is a subspace of V.
  - (c)  $0 \in W$  since f(x) = 0 for all x implies f'(x) = f''(x) = 0 for all x and hence that f''(x) + xf'(x) + f(x) = 0 for all x. If  $a, b \in \mathbb{F}, f, f \in W$  then (af + bg)''(x) + x(af + bg)'(x) + (af + bg)(x) = af''(x) + bg''(x) + x(af'(x) + bg'(x)) + af(x) + bg(x) = a(f''(x) + xf'(x) + f(x)) + b(g''(x) + xg''(x) + g(x)) = 0 so that W is a subspace of V. Note that if  $f \in \mathbb{C}^{\mathbb{R}}$  then  $f = f_1 + if_2$  with  $f_1, f_2 \in \mathbb{R}^{\mathbb{R}}$  and  $f' = f'_1 + if'_2$ . If  $a = a_1 + ia_2, b = b_1 + ib_2$  with  $a_i, b_i \in \mathbb{R}$  and  $g = g_1 + ig_2$  with  $g_1, g_2 \in \mathbb{R}^{\mathbb{R}}$  we have  $af + bg = (a_1f_1 a_2f_2 + b_1g_1 b_2g_2) + i(a_1f_2 + a_2f_1 + b_1g_2 + b_2g_1)$  which implies that (af + bg)' = af' + bg'.
  - (d) If  $x, y \in \mathbb{R}$  with  $x^2 xy + y^2 = 0$  then x = y = 0 (complete the square). Hence  $W = \{(a, b, 0) \mid a, b \in \mathbb{R}\}$  which is a subspace of V if  $\mathbb{F} = \mathbb{R}$ . If  $\mathbb{F} = \mathbb{C}$  then W is not a subspace since  $(1, 0, 0) \in W$  but  $i(1, 0, 0) = (i, 0, 0) \notin W$  since  $i \notin \mathbb{R}$ .
  - (e) The zero matrix 0 is in W since  $0^t = \overline{0} = 0$ . If  $X, Y \in W$  then  $(X + Y)^t = X^t + Y^t = \overline{X} + \overline{Y} = \overline{X + Y}$  which implies that  $X + Y \in W$ . If  $\mathbb{F} = \mathbb{R}$  and  $a \in \mathbb{R}, X \in W$  then  $\overline{aX} = a\overline{X} = aX$  which implies that W is a subspace of V if  $\mathbb{F} = \mathbb{R}$ . If  $\mathbb{F} = \mathbb{C}$  we have  $X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  is in W but  $iX = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$  is not so that W is not a subspace of V in this case.
- 3. (a) We have  $f \in V_{\text{even}} \iff f(-x) = f(x)$  for all x and  $f \in V_{\text{odd}} \iff f(-x) = -f(x)$  for all x. For any  $f \in V$  we have f = g + h where g(x) = (f(x) + f(-x))/2, h(x) = (f(x) f(-x))/2. Since g is even and h is odd we have  $V = V_{\text{even}} + V_{\text{odd}}$ . The sum is direct since  $V_{\text{even}} \cap V_{\text{odd}} = \{0\}$  since f(x) = f(-x), f(-x) = -f(x) implies -f(x) = f(x) which implies f(x) = 0.
  - (b) We have  $\exp(x) = e^x = (e^x + e^{-x})/2 + (e^x e^{-x})/2 = \cosh(x) + \sinh(x)$ so that  $\exp = \cosh + \sinh x$  is the decomposition of the exponential function exp as the sum of an even function and an odd function.
- 4. (a) Let  $W = \{v \in V \mid v \in W_i \text{ for all } x \in I\}$  be the intersection of the family of subspaces  $W_i$   $(i \in I)$ . The Then  $0_V \in W$  since  $0_V \in W_i$  for all *i*. If *a*, *b* are scalars and  $u, v \in W$  then  $u, v \in W_i$  for all *i* which implies that  $au + bv \in W_i$  for all *i* and hence that  $au + bv \in W$ .
  - (b) The statement is false since  $\mathbb{R}^2 = \mathbb{R}(1,0) \oplus \mathbb{R}(0,1) = \mathbb{R}(1,0) \oplus \mathbb{R}(1,1)$  but  $\mathbb{R}(0,1) \neq \mathbb{R}(1,1)$ .