McGill University
MATH 236 Midterm Test

Justify All Your Assertions
Determinants not allowed

1. Determine which of the following subsets S of the given real vector space V are subspaces of V.

   (a) $V = \mathbb{R}^{\mathbb{R}}$, $S = \{f \in V \mid f(2-x) = xf(x) \text{ for all } x \in \mathbb{R}\}$.

   (b) $V = \mathbb{R}^2$, $S = \{(x, y) \in V \mid x^2 = y^2\}$.

2. Determine whether the following sequences in $\mathbb{R}^{\mathbb{R}}$ are linearly independent or linearly dependent.

   (a) $(x-1)^2$, $(x+1)^2$, $(x-2)^2$, $(x+2)^2$.

   (b) e^x, e^{2x}, e^{3x}.

3. (a) Define the following terms: kernel of a linear mapping, eigenvalue of a linear operator.

   (b) If T is a linear operator on a vector space V and a, b are scalars, show that $T(v) = \lambda v$ implies that $(T^2 + aT + b)(v) = (\lambda^2 + a\lambda + b)v$. If $v \neq 0$ and $T^2 + aT + b = 0$ deduce that $\lambda^2 + a\lambda + b = 0$.

4. Let $V = \mathbb{R}^3$ and let $T : V \to V$ be defined by

$$T(x, y, z) = (2x + y + z, x + 2y + z, x + y + 2z).$$

   (a) Show that T is linear and that $T^2 - 5T + 4 = 0$.

   (b) Show that $V = \text{Ker}(T-1) \oplus \text{Ker}(T-4)$.

   (c) Find the eigenvalues of T and a basis of V consisting of eigenvectors of T.