

McGill University  
Math 236: Algebra 2  
Assignment 6: due Monday, April 10, 2006

1. (a) Find the spectral decomposition of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

**Hint:** By inspection 3 and 5 are eigenvalues of  $A$ . Explain why this is the case.

- (b) Using (a), solve the initial value problem

$$i \frac{dX}{dt} = Ax, \quad X(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

- (c) Using (a), find the maximum and minimum values of the quadratic form

$$q(X) = 3x_1^2 + 2x_1x_2 + 2x_1x_3 + 3x_2^2 + 2x_2x_4 + 3x_3^2 + 2x_3x_4 + 3x_4^2$$

subject to the constraint  $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$  and find where these maxima and minima are attained.

2. Find the Jordan canonical form of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and find an invertible matrix such that  $P^{-1}AP$  is in Jordan canonical form.

3. **(Bonus Question)** Find the spectral decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

4. **(Bonus Question)**

- (a) Prove or give a counterexample: the product of any two self-adjoint operators on an inner-product space is self-adjoint.
- (b) If  $T$  is a normal operator on a finite-dimensional complex inner product space such that  $T^9 = T^8$  prove that  $T$  is self-adjoint and  $T^2 = T$ .

5. **(Bonus Question)**

- (a) Let  $T$  be a linear operator on a vector space  $V$  such that  $\dim(\text{Im}(T)) = n$ . Prove that  $T$  has at most  $n + 1$  distinct eigenvalues.
- (b) If  $T$  is a linear operator on a vector space  $V$  such that any vector of  $V$  is an eigenvector of  $T$  prove that  $T$  is a scalar multiple of the identity operator.