McGill University Math 247: Algebra 2 Assignment 5: due Wednesday, March 29, 2006

1. The trace of a square matrix $A = [a_{ij}] \in F^{n \times n}$ is the scalar

$$tr(A) = \sum_{i=1}^n a_{i\,i} = a_{11} + \ldots + a_{nn}.$$

- (a) Show that tr is linear and that tr(AB) = tr(BA).
- (b) If $\phi : F^{n \times n} \to F$ is linear and $\phi(AB) = \phi(BA)$ for all $A, B \in F^{n \times n}$, show that $\phi = c$ tr for some scalar c. **Hint:** Show that the subspace spanned by the matrices of the form AB BA has dimension $n^2 1$.
- 2. (a) Show that $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 5x_2y_2$ defines an inner product on \mathbb{R}^2 .
 - (b) Show that $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 \bar{y}_1 + i x_1 \bar{y}_2 i x_2 \bar{y}_1 + 2 x_2 \bar{y}_2$ defines an inner product on \mathbb{C}^2 .
- 3. Let $V = C_{\mathbb{R}}([0,1])$ with inner product $\langle f,g \rangle = \int_0^1 f(x)g(x) dx$.
 - (a) Find the best approximation to $f(x) = \sin(\pi x)$ by a function in Span $(1, x, x^2)$.
 - (b) If ϕ is the linear form on $W = \text{Span}(1, \mathbf{x}, \mathbf{x}^2)$ defined by $\phi(f) = f(1)$, find $g \in W$ such that $\phi(f) = \langle f, g \rangle$ for all $f \in W$.
- 4. Let $V = \mathbb{C}^{n \times n}$ with the inner product $\langle X, Y \rangle = \operatorname{tr}(X\overline{Y}^{t})$ and let T be the linear operator on V defined by

$$T(X) = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} X - X \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}.$$

- (a) Show that T is self-adjoint.
- (b) Find an orthonormal basis of V consisting of eigenvectors of T. Hint: Use the fact that $Im(T)^{\perp} = Ker(T)$.
- (c) Find the matrix A of T with respect to the standard basis of V and find a unitary matrix U such that $U^{-1}AU$ is a diagonal matrix.