McGill University
Math 236: Algebra 2
Assignment 1: due Friday, January 20, 2006

1. (a) Show that the set $\mathbb{R}^2$, together with the operations $\oplus, \odot$ defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2 - 1, y_1 + y_2 + 2), \quad a \odot (x, y) = (ax + 1 - a, ay - 2 + 2a),$$

is a vector space over $\mathbb{R}$.

   (b) Show that this vector space is isomorphic to $\mathbb{R}^2$ under the usual operations. **Hint:** Consider the mapping T defined by $T(x, y) = (x - 1, y + 2)$.

2. In each of the following decide whether or not W is a subspace of the vector space V over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

   (a) $V = \mathbb{F}^\infty$, $W = \{ (x_1, \ldots, x_n, \ldots) \in V \mid x_{n+3} = x_{n+2} + nx_n \quad \text{for} \quad n \geq 1\}$;

   (b) $V = \mathbb{F}^{\mathbb{R}}$, $W = \{ f \in V \mid f(x) = f(x^2 + 1) \text{ for all } x \in \mathbb{R}\}$;

   (c) $V = \mathbb{F}^{\mathbb{R}}$, $W = \{f \in V \mid f''(x) + xf'(x) + f(x) = 0\}$ where $f'(x)$ is the derivative of f at x;

   (d) $V = \mathbb{F}^3$, $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 - x_1x_2 + x_2^2 = 0\}$;

   (e) $V = \mathbb{C}^{2\times 2}$, $W = \{X \in V \mid X^t = \overline{X}\}$ where X^t is the transpose of X and $\overline{X}$ is the conjugate of X, the matrix obtained from X by replacing each entry by its complex conjugate)

3. (a) Let V be the real vector space $\mathbb{R}^{\mathbb{R}}$ and let V_{even}, V_{odd} be the subsets of V consisting the even and the odd functions respectively. Show that V_{even}, V_{odd} are subspaces of V and that $V = V_{\text{even}} \oplus V_{\text{odd}}$.

   (b) Using (a), give the decomposition of the function $f(x) = e^x$ into its even and odd component functions.

4. If V is a vector space, prove or disprove the following statements:

   (a) The intersection of any family of subspaces $(W_i)_{i\in I}$ of V is a subspace of V.

   (b) If U_1, U_2, W are subspaces of V with $U_1 \oplus W = U_2 \oplus W$ then $U_1 = U_2$.