McGill University
Math 223 B: Linear Algebra
Sample Midterm Exam

Attempt all Questions.
All questions are of equal value.
No Calculators Allowed.

1. Show that $x = (5t - 3s, 3t - 2s, s, 2t, -2t)$ is a parametric description of the solution set of the homogeneous system of linear equations

$$\begin{aligned} 3x_1 - 5x_2 - 1x_3 + 4x_4 + 4x_5 &= 0 \\ -2x_1 + 4x_2 + 2x_3 + 7x_4 + 8x_5 &= 0 \\ 5x_1 - 9x_2 - 3x_3 - 3x_4 - 4x_5 &= 0 \\ -2x_1 + 6x_2 + 6x_3 + 5x_4 + 9x_5 &= 0, \end{aligned}$$

given that the rank of the coefficient matrix is 3. Do not solve this system of equations using Gaussian elimination. State any general fact about matrices and vector spaces that you use.

2. In each case, tell whether the given set V is a vector space or not; justify your answers.

   (a) V is the set of functions from $\mathbf{R}$ to $\mathbf{R}$ such that $f'(x) = xf(x^2)$ for any $x \in \mathbf{R}$;

   (b) $V = \{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{R}^{2\times 2} \mid A^2 = A\}$.

3. (a) Show that $\{\cos x, \cos 2x, \cos 3x\}$ is a linearly independent set of functions.

   (b) Show that $(1+x)^3, (1-x)^3, (1+2x)^3, (1-2x)^3, (1+3x)^3$ are linearly dependent functions. State clearly any general fact about vector spaces that you use to justify your assertion. Hint: First show that these functions are in the subspace spanned by $1, x, x^2, x^3$.

4. Find a basis for the subspace of $\mathbf{R}^{2\times 2}$ generated by the matrices

$$\begin{bmatrix} 5 & 1 \\ -2 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} -4 & 0 \\ 4 & -4 \end{bmatrix}, \begin{bmatrix} 16 & 5 \\ -1 & 11 \end{bmatrix},$$

and complete this basis to a basis of $\mathbf{R}^{2\times 2}$.