McGill University
Math 223B: Linear Algebra
Solution Sheet for Assignment 6

1. a) $T(1) = 0$, which has coordinates $\begin{bmatrix}0\\0\\0\end{bmatrix}$; $T(x) = 1 - x$, which has coordinates $\begin{bmatrix}1\\-1\\0\end{bmatrix}$; $T(x^2) = 2x - 2x^2$, which has coordinates $\begin{bmatrix}2\\-2\\0\end{bmatrix}$. Thus $M_B(T) = \begin{bmatrix}0 & 1 & 0\\0 & -1 & 2\\0 & 0 & -2\end{bmatrix}$.

   b) $M_B(T)$ has characteristic polynomial $x(x+1)(x+2)$ and hence eigenvalues 0, -1 and -2.
   Eigenspaces (in coordinates):

$$E_{-2} = null\begin{pmatrix}-2 & -1 & 0\\0 & -1 & -2\\0 & 0 & 0\end{pmatrix} = \left\{\begin{bmatrix}t\\-2t\\t\end{bmatrix}\right\} = span\left\{\begin{bmatrix}1\\-2\\1\end{bmatrix}\right\}.$$

   So $E_{-2} = span\{1 - 2x + x^2\}$. Similarly $E_{-1} = span\{1 - x\}$ and $E_0 = span\{1\}$.
   The kernel of T is the eigenspace with eigenvalue 0. The image of T (in coordinates) is the column space of $M_B(T)$. So $im(T) = span\{1 - x, x - x^2\}$.

2. a) Not an inner product, since not symmetric (the corresponding matrix is $\begin{pmatrix}2 & 2\\0 & 1\end{pmatrix}$, or for example $\langle(1,0),(0,1)\rangle = 2$, but $\langle(0,1),(1,0)\rangle = 0$).

   b) Not an inner product, since not linear in the first argument. For example $\langle(1,0),(1,0)\rangle = 4$ but $\langle 2(1,0),(1,0)\rangle = 10 \neq 2\langle(1,0),(1,0)\rangle$.

   c) This is an inner product.

$$\langle(x_1,x_2),(y_1,y_2)\rangle = (x_1,x_2)A\begin{pmatrix}y_1\\y_2\end{pmatrix}$$

   with $A = \begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix}$. So $\langle\ ,\ \rangle$ is linear in both arguments. Also $\langle\ ,\ \rangle$ is symmetric since A is symmetric.
   Completing the squares (starting with x_2) we obtain $\langle(x_1,x_2),(x_1,x_2)\rangle = 2x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2 + x_1^2 \geq 0$. If this is 0, then $x_1 = 0$ and $x_1 - x_2 = 0$, so $x_1 = x_2 = 0$.
   Other possible proofs for positive definiteness are showing that all eigenvalues of A are positive or looking at the $det({}^rA)$ (compare question 3).

   d) This is linear in both arguments and symmetric, but not positive definite and hence not an inner product. If we complete the square, we have $\langle(x_1,x_2),(x_1,x_2)\rangle = x_1^2 + 2x_1x_2 + x_2^2 = (x_1 + x_2)^2$; so we see that for example $\langle(1,-1),(1,-1)\rangle = 0$. Other possible proofs that $\langle\ ,\ \rangle$ is not positive definite: The corresponding matrix $A = \begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$ has an eigenvalue $\lambda \leq 0$. Or look at the ${}^rA$; for example $det({}^2A) = det(A) = 0 \leq 0$.

3. a)

$$\langle(x_1,x_2,x_3),(y_1,y_2,y_3)\rangle = (x_1,x_2,x_3)A\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix} \text{ with } A = \begin{pmatrix}1 & 1 & 0\\1 & 6 & 2\\0 & 2 & 1\end{pmatrix}.$$

   So $\langle\ ,\ \rangle$ is linear in both arguments and symmetric (since A is symmetric).
   There are 3 possible ways to show that $\langle\ ,\ \rangle$ is positive definite.
   First possibility: Completing the squares we obtain $\langle(x_1,x_2,x_3),(x_1,x_2,x_3)\rangle = x_1^2 + 2x_1x_2 + 6x_2^2 + 4x_2x_3 + x_3^2 = (x_1 + x_2)^2 + 5x_2^2 + 4x_2x_3 + x_3^2 = (x_1 + x_2)^2 + 5(x_2 + \frac{2}{5}x_3)^2 + \frac{1}{5}x_3^2 \geq 0$. If this is 0, then $x_3 = 0$, $x_2 + \frac{2}{5}x_3 = 0$ and $x_1 + x_2 = 0$, so $x_1 = x_2 = x_3 = 0$.
   Second possibility: A has charateristic polynomial $(x-1)(x^2 - 7x + 1)$. The eigenvalues of A, namely 1 and $\frac{7\pm\sqrt{45}}{2}$ are all positive, hence A is positive definite.
   Third possibility: We have

$$det({}^1A) = 1 > 0, \quad det({}^2A) = \begin{vmatrix}1 & 1\\1 & 6\end{vmatrix} = 5 > 0 \text{ and } det({}^3A) = \begin{vmatrix}1 & 1 & 0\\1 & 6 & 2\\0 & 2 & 1\end{vmatrix} = 1 > 0,$$

   hence A is positive definite.

b) First we have to find an orthogonal basis $\{E_1, E_2\}$ of U. Orthogonalization by Gram-Schmidt: Let $X_1 = (1, 0, -1)$ and $X_2 = (0, 1, 3)$. Take $E_1 = X_1$ and

$$E_2 = X_2 - \frac{\langle X_2, E_1 \rangle}{\langle E_1, E_1 \rangle} E_1 = (0, 1, 3) - \frac{-4}{2}(1, 0, -1) = (2, 1, 1).$$

Then

$$proj_U(Z) = \frac{\langle Z, E_1 \rangle}{\langle E_1, E_1 \rangle} E_1 + \frac{\langle Z, E_2 \rangle}{\langle E_2, E_2 \rangle} E_2 = \frac{-5}{2}(1, 0, -1) + \frac{38}{19}(2, 1, 1) = (\frac{3}{2}, 2, \frac{9}{2}).$$

4. We construct an orthogonal basis $\{E_1, E_2, E_3\}$ by Gram-Schmidt:

$$\begin{aligned} E_1 &= 1. \\ E_2 &= x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} 1 = x^2 - \frac{1}{3}. \\ E_3 &= x^4 - \frac{\langle x^4, 1 \rangle}{\langle 1, 1 \rangle} 1 - \frac{\langle x^4, E_2 \rangle}{\langle E_2, E_2 \rangle} E_2 = x^4 - \frac{1}{5} - \frac{6}{7}(x^2 - \frac{1}{3}) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}. \end{aligned}$$

Among others we have used the following calculations:

$$\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3} \rangle = \int_0^1 x^4 - \frac{2}{3}x^2 + \frac{1}{9} dx = \frac{1}{5} - \frac{2}{9} + \frac{1}{9} = \frac{4}{45} \text{ and } \langle x^4, x^2 - \frac{1}{3} \rangle = \int_0^1 x^6 - \frac{1}{3}x^4 dx = \frac{8}{105}.$$