McGill University
Math 223B: Linear Algebra
Solution Sheet for Assignment 5

1. The characteristic polynomial of A is $\begin{vmatrix} \lambda-4 & -2 \\ 1 & \lambda-2 \end{vmatrix} = \lambda^2 - 6\lambda + 10$. The roots of this polynomial are $3 \pm i$; these are the eigenvalues of A. The eigenspace for the eigenvalue $3+i$ is the solution space of the system $(-1+i)x-2y = x+(1-i)y = 0$. This system is equivalent to the single equation $x+(1+i)y = 0$ and so the eigenspace for the eigenvalue $3-i$ is the subspace of $\mathbb{C}^2$ spanned by $(-1-i,1)$. Since A is a real matrix, the eigenspace corresponding to the eigenvalue $3-i$, the conjugate of $3+i$, is the span of the vector $(-1+i,1)$, the conjugate of $(-1-i,1)$. The eigenvectors $(-1-i,1),(-1+i,1)$ are linearly independent and so, if $P = \begin{bmatrix} -1-i & -1+i \\ 1 & 1 \end{bmatrix}$, we have $P^{-1}AP = \begin{bmatrix} 3+i & 0 \\ 0 & 3-i \end{bmatrix}$.

2. (a) The characteristic polynomial of A is $(\lambda-1)^3(\lambda-5)$. The eigenspace for the eigenvalue 1 is the solution space of the single equation $x+y+z+w=0$. Applying the Gram-Schmidt process to basis $(-1,1,0,0),(-1,0,1,0),(-1,0,0,1)$ of this eigenspace, we get the vectors

$$u = (-1/\sqrt{2}, 1/\sqrt{2}, 0, 0),\ v = (1/\sqrt{6}, 1/\sqrt{6}, -2/\sqrt{6}, 0),\ w = (1/\sqrt{12}, 1/\sqrt{12}, 1/\sqrt{12}, -3/\sqrt{12}).$$

The eigenspace of A corresponding to the eigenvalue 5 is the span of the vector $z = (1/2, 1/2, 1/2, 1/2)$. If P is the matrix whose columns are the transposes of these row vectors, then P is orthogonal and $P^{-1}AP = \operatorname{diag}(1,1,1,5)$.

(b) The required matrices Q, R are

$$Q = u^tu + v^tv + w^tw = \begin{bmatrix} 3/4 & -1/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & -1/4 & 3/4 \end{bmatrix}, \quad R = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}.$$

We have $A^n = Q + 5^nR$.

(c) We have $B = Q + \sqrt{5}R$.

3. (a) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. Then $T(aX+bY) = A(aX+bY) - (aX+bY)A = aAX + bAY - aXA - bYA = a(AX-XA) + b(AY-YA) = aT(X) + bT(Y)$. Hence T is linear.

(b) If $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have $T(X) = \begin{bmatrix} 2c-2b & -2a-2b+2d \\ 2a+2c-2d & 2b-2c \end{bmatrix}$. Hence $T(X) = 0$ iff $b-c = a+b-d = a+c-d = 0$. This system, in Gauss reduced form, is $a+c-d = b-c = 0$. Hence $T(X) = 0$ iff

$$X = \begin{bmatrix} d-c & c \\ c & d \end{bmatrix} = c\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ form a basis for the kernel of T. Since $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ can be used to complete the above basis of the kernel of T to a basis of $\mathbb{R}^{2\times 2}$, the vectors $T(A) = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}, T(B) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ form a basis for the image of T.

(c) The rank and nullity of T are both 2. Hence T is neither onto nor one-to-one, in particular, it is not an isomorphism.

4. (a) Since $T(E_1) = -2E_2 + E_3$, $T(E_2) = -2E_1 - 2E_2 + 2E_3$, $T(E_3) = 2E_1 + 2E_3 - 2E_4$, $T(E_4) = 2E_2 - 2E_3$, the matrix of T with respect to the basis E_1, E_2, E_3, E_4 is the matrix

$$A = \begin{bmatrix} 0 & -2 & 2 & 0 \\ -2 & -2 & 0 & 2 \\ 2 & 0 & 2 & -2 \\ 0 & 2 & -2 & 0 \end{bmatrix}.$$

(b) Since $T(F_1) = 0$, $T(F_2) = 2F_1 - 2F_2 + 2F_3 - 2F_4$, $T(F_3) = 0$, $T(F_4) = 6F_1 - 8F_2 + 2F_3 + 2F_4$, the matrix of T with respect to the basis F_1, F_2, F_3, F_4 is the matrix

$$B = \begin{bmatrix} 0 & 2 & 0 & 6 \\ 0 & -2 & 0 & -8 \\ 0 & 2 & 0 & 2 \\ 0 & -2 & 0 & 2 \end{bmatrix}.$$

(c) The required matrix is $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$.

5. Let $f_1 = (1,2,1,2), f_2 = (1,0,1,1), f_3 = (0,0,1,0), f_4 = (0,0,0,1)$. These vectors form a basis of $\mathbb{R}^4$. If (y_1, y_2, y_3, y_4) is the coordinate vector of $v \in \mathbb{R}^4$ with respect to this basis, we have $v = y_1 f_1 + y_2 f_2 + y_3 f_3 + y_4 f_4$. We define a linear mapping of $\mathbb{R}^4$ into itself by $T(v) = y_3 f_1 + y_4 f_4$. This is the linear mapping with $T(f_1) = T(f_2) = 0$, $T(f_3) = f_1$, $T(f_4) = f_2$ and hence the kernel and image of T are spanned by f_1, f_2. The required matrix is therefore the matrix of T with respect to the standard basis e_1, e_2, e_3, e_4 of $\mathbb{R}_4$. Now $e_1 = f_2 - f_3 - f_4, 2e_2 = f_1 - f_2 - f_4, e_3 = f_3, e_4 = f_4$ and so $T(e_1) = -f_1 - f_2 = (-2,-2,-2,-3)$, $T(e_2) = -f_2/2 = (-1/2, 0, -1/2, -1/2)$, $T(e_3) = f_1 = (1,2,1,2)$, $T(e_4) = f_2 = (1,0,1,1)$. Hence the matrix is

$$\begin{bmatrix} -2 & -1/2 & 1 & 1 \\ -2 & 0 & 2 & 0 \\ -2 & -1/2 & 1 & 1 \\ -3 & -1/2 & 2 & 1 \end{bmatrix}$$

has the required properties.