McGill University
Math 223B: Linear Algebra
Solution Sheet for Assignment 4

1.

$$c_A(x) = |xI - A| = \begin{vmatrix} x-2 & 2 & -1 & -1 \\ 0 & x-3 & 0 & 0 \\ 1 & 2 & x-4 & -1 \\ -1 & 2 & 2 & x-1 \end{vmatrix} \overset{2.\ row}{=} (x-3) \begin{vmatrix} x-2 & -1 & -1 \\ 1 & x-4 & -1 \\ -1 & 2 & x-1 \end{vmatrix} R_3 = R_3 + R_2$$

$$= (x-3) \begin{vmatrix} x-2 & -1 & -1 \\ 1 & x-4 & -1 \\ 0 & x-2 & x-2 \end{vmatrix} C_2 = C_2 - C_3 \quad = (x-3) \begin{vmatrix} x-2 & 0 & -1 \\ 1 & x-3 & -1 \\ 0 & 0 & x-2 \end{vmatrix} = (x-3)^2(x-2)^2.$$

So the eigenvalues are 3 and 2.
Eigenspaces:

$$E_3 = \mathrm{null}(3I_4 - A) = \mathrm{null} \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & -1 & -1 \\ -1 & 2 & 2 & 2 \end{pmatrix} \overset{Gauss}{=} \mathrm{null} \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{span} \left\{ \begin{pmatrix} 6 \\ -1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 0 \\ 4 \end{pmatrix} \right\}.$$

Similarly:

$$E_2 = \mathrm{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

2. If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, then $f'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + 2a_2x + a_1$ and $xf'(x) = na_nx^n + (n-1)a_{n-1}x^{n-1} + \ldots + 2a_2x^2 + a_1x$. We see that ϕ is not onto, because there is no $f \in V$ with $\phi(f) = 1$. Also ϕ is not one-to-one, because $\phi(x) = x = \phi(x+1)$.

   Let
   $$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$$
   (with $a_n \neq 0$) be an eigenvector with eigenvalue λ. Then
   $$\lambda a_n x^n + \lambda a_{n-1}x^{n-1} + \ldots + \lambda a_2x^2 + \lambda a_1 x + \lambda a_0$$
   $$= \phi(f) = na_nx^n + (n-1)a_{n-1}x^{n-1} + \ldots + 2a_2x^2 + a_1x.$$
   So
   $$\lambda a_n = na_n,\ \lambda a_{n-1} = (n-1)a_{n-1},\ \ldots,\ \lambda a_1 = a_1,\ \lambda a_0 = 0.$$
   As $a_n \neq 0$, by the first equation we have $\lambda = n$. The other equations imply $a_i = 0$ for $i < n$. So $f(x) = a_nx^n$.
   Result: The eigenvalues of ϕ are $\lambda = 0,\ 1,\ 2,\ 3, \ldots$ with corresponding eigenspaces $E_n = span\{x^n\}$.

3. a) A an B are not similar because they have different traces (or different determinants or different characteristic polynomials).

   b) A an B are not similar because they have different determinants (or different characteristic polynomials).

   c) A and B have the same trace, the same determinants, the same characteristic polynomial, the same rank. We suspect that they are similar.
   Smart proof: $c_A(x) = (x-5)(x-4)$. All eigenvalues are real and have multiplicity one, so A is diagonalizable, i.e. there exists an invertible matrix P with $P^{-1}AP = \left(\begin{smallmatrix} 5 & 0 \\ 0 & 4 \end{smallmatrix}\right)$. By the same argument there exists an invertible matrix Q with $Q^{-1}BQ = \left(\begin{smallmatrix} 5 & 0 \\ 0 & 4 \end{smallmatrix}\right)$. So $B = QP^{-1}APQ^{-1} = (PQ^{-1})^{-1}A(PQ^{-1})$.
   Other proof: Write $P = \left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)$. Then $AP = PB$ is equivalent to the system of linear equations

   $$\begin{aligned} 5x_1 + x_3 &= 5x_1 + x_2 \\ 5x_2 + x_4 &= 4x_4 \\ 4x_3 &= 5x_3 + x_4 \\ 4x_4 &= 4x_4 \end{aligned}$$

   We need a solution such that P is invertible. For example $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. Then $B = P^{-1}AP$.

4.

$$c_A(x) = \begin{vmatrix} x+7 & 9 \\ -6 & x-8 \end{vmatrix} = x^2 - x - 2 = (x-2)(x+1).$$

$E_2 = \text{null}(2I_2 - A) = \text{null}\left(\begin{smallmatrix} 9 & 9 \\ -6 & -6 \end{smallmatrix}\right) = \text{null}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ by Gauss elimination. So $E_2 = \text{span}\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$.

Analoguously $E_1 = \text{null}(-I_2 - A) = \text{null}\left(\begin{smallmatrix} 6 & 9 \\ -6 & -9 \end{smallmatrix}\right) = \text{null}\left(\begin{smallmatrix} 2 & 3 \\ 0 & 0 \end{smallmatrix}\right)$, so $E_{-1} = \text{span}\left(\begin{smallmatrix} 3 \\ -2 \end{smallmatrix}\right)$.

Let $P = \left(\begin{smallmatrix} 1 & 3 \\ -1 & -2 \end{smallmatrix}\right)$. Then P is invertible and

$$P^{-1}AP = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

Moreover

$$A^n = (P\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}P^{-1})^n = P\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}^n P^{-1} = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^n & 3(-1)^n \\ -2^n & -2(-1)^n \end{pmatrix}\begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -2^{n+1} + 3(-1)^n & -3 \cdot 2^n + 3(-1)^n \\ 2^{n+1} - 2(-1)^n & 3 \cdot 2^n - 2(-1)^n \end{pmatrix}.$$

5. Let

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 6 \end{pmatrix}.$$

Orthogonalization by Gram-Schmidt:

$$E_1 = X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad E_2 = X_2 - \frac{X_2 \bullet E_1}{||E_1||^2}E_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{2}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{-1}{2} \\ \frac{-1}{2} \\ \frac{-1}{2} \end{pmatrix},$$

$$E_3 = X_3 - \frac{X_3 \bullet E_1}{||E_1||^2}E_1 - \frac{X_3 \bullet E_2}{||E_2||^2}E_2 = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 6 \end{pmatrix} - \frac{4}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{-8}{3}\begin{pmatrix} \frac{3}{2} \\ \frac{-1}{2} \\ \frac{-1}{2} \\ \frac{-1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-4}{3} \\ \frac{-7}{3} \\ \frac{11}{3} \end{pmatrix}.$$

Then $\{E_1, E_2, E_3\}$ is an orthogonal basis of U. To obtain an orthonormal basis we have to divide the vectors by their lengths: $||E_1|| = 2$, $||E_2|| = \sqrt{3}$, $||E_3|| = \frac{1}{3}\sqrt{186}$. So orthonormal basis:

$$\left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{3}{2\sqrt{3}} \\ \frac{-1}{2\sqrt{3}} \\ \frac{-1}{2\sqrt{3}} \\ \frac{-1}{2\sqrt{3}} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{-4}{\sqrt{186}} \\ \frac{-7}{\sqrt{186}} \\ \frac{11}{\sqrt{186}} \end{pmatrix} \right\}.$$