McGill University
Math 223B: Linear Algebra
Solution Sheet for Assignment 3

1. Bringing A to reduced echelon form, we find the matrix

$$B = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since A is row equivalent to B, the row space of A is equal to the row space of B. Since the non-zero rows of a matrix in echelon are a basis for the row space of that matrix, the vectors

$$(1, 0, -4, -28, -37, 13),\ (1, -2, -12, -16, 5)$$

constitute a basis for the row space of A. Also, since A is row equivalent to B, the null space of A is equal to the null space of B and hence is equal to the solution set of the system

$$\begin{aligned} x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 &= 0 \\ x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 &= 0 \end{aligned}$$

. The solutions of this system are the vectors

$$x = t_1(4, 2, 1, 0, 0, 0) + t_2(28, 12, 0, 1, 0, 0) + t_3(37, 16, 0, 0, 1, 0) + t_4(-13, -5, 0, 0, 0, 1)$$

with t_1, t_2, t_3, t_4 arbitrary scalars. Since x=0 if and only if $t_1 = t_2 = t_3 = t_4 = 0$, the vectors

$$(4, 2, 1, 0, 0, 0),\ (28, 12, 0, 1, 0, 0),\ (37, 16, 0, 0, 1, 0),\ (-13, -5, 0, 0, 0, 1)$$

form a basis for the solution space of A. Since $x = (x_1, x_2, x_3, x_4, x_5, x_6)$ is in the null space of A if and only if

$$x_1 \begin{bmatrix} -1 \\ 1 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -3 \\ -8 \\ -9 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 8 \\ 12 \\ -4 \end{bmatrix} + x_5 \begin{bmatrix} 5 \\ 11 \\ 17 \\ -4 \end{bmatrix} + x_6 \begin{bmatrix} -3 \\ -2 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and there are solutions with (a) $x_3 = 1, x_4 = x_5 = x_6 = 0$, (b) $x_4 = 1, x_3 = x_5 = x_6 = 0$, (c) $x_5 = 1, x_3 = x_4 = x_6 = 0$, (d) $x_6 = 1, x_3 = x_4 = x_5 = 0$, we see that the last four columns of A are in the subspace spanned by the first two columns. Since the first two columns are linearly independent ($x_3 = x_4 = x_4 = x_6 = 0$ implies $x_1 = x_2 = 0$), they form a basis for the column space of A. The rank of A is 2, being equal to the common dimension of the row and column spaces of A, and the nullity of A is 4, being the dimension of the null space of A.

2. Using Gaussian elimination we find that the given system has the same solution set W as the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 2x_3 + x_4 &= 0 \\ 6x_3 - 5x_4 + 2x_5 + x_6 &= 0. \end{aligned}$$

Thus the mapping $T : W \to \mathbb{R}^3$ defined by $T(x) = (x_4, x_5, x_6)$ is an isomorphism of vector spaces. This shows that $\dim(W) = 3$. Verifying that the given vectors are in W and that they are linearly independent, we deduce that they must be a basis for W since any n linearly independent vectors in an n-dimensional vector space V are a basis of V.

3. The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in W if and only if

$$x_1 \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 & -5 \\ -6 & 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix} + x_4 \begin{bmatrix} 1 & -6 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for some scalars x_1, x_2, x_3, x_4. This is equivalent to the consistency of the system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + x_4 &= a \\ -5x_1 - 5x_2 - 4x_3 - 6x_4 &= b \\ -4x_1 - 6x_2 - 5x_3 - 5x_4 &= c \\ 2x_1 + 6x_2 + 7x_3 + x_4 &= d. \end{aligned}$$

By Gaussian elimination, this system has the same solution set as the system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + x_4 &= a \\ x_2 + x_4 &= a + b - c \\ 3x_3 - 3x_4 &= 2a - 2b + 3c \\ 0 &= 6a + c - d. \end{aligned}$$

From this it follows that $\dim(W) = 3$, that the first three matrices form a basis of W and that this basis can be completed to a basis of V by adding the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. We also obtain that W is the set of those matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $6a + c - d = 0$. If we also require that the trace $a + d$ be zero then the subspace Z of W consisting of those matrices of trace 0 consists of those matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $6a + c - d = a + d = 0$ which is equivalent to $a = -d,\ c = 7d$, i.e.,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -d & b \\ 7d & d \end{bmatrix} = d \begin{bmatrix} -1 & 0 \\ 7 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Hence Z has basis $\begin{bmatrix} -1 & 0 \\ 7 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

4. (a) The zero sequence 0 is in W since $0_n = 0$ for all n and so $0_{n+2} = 0 = 15 \cdot 0_n - 2 \cdot 0_{n+1}$. If $x, y \in W$ and $a, b \in \mathbb{R}$ we have $(ax + by)_{n+2} = ax_{n+2} + by_{n+2} = a(15x_n - 2x_{n+1}) + b(15y_n - 2y_{n+1}) = 15(ax_n + by_n) - 2(ax_{n+1} + by_{n+1}) = 15(ax + by)_n - 2(ax + by)_{n+1}$ which shows that $ax + by \in W$. Hence W is a subspace of V.

   (b) We have to show that T is linear, one-to-one and onto. Now T is linear since $T(ax + by) = (ax_0 + by_0, ax_1 + by_1) = a(T(x) + bT(y)$. If $T(x) = T(y)$ we must have $x = y$ since a sequence in W is completely determined by the first two terms. Thus T is one-to-one. If $(a, b) \in \mathbb{R}^2$, we can inductively define an x by $x_0 = a, x_1 = b$ and $x_{n+2} = 15x_n - x_{n+1}$ for $n > 0$. This sequence is in W by definition and $T(x) = (a, b)$. Hence T is onto. Since isomorphic vector spaces have the same dimension, we have $\dim(W) = \dim(\mathbb{R}^2) = 2$.

   (c) Since $T(u) = (1, 3)$, $T(v) = (1, -5)$ and $(1, 3), (1, -5)$ are a basis of $\mathbb{R}^2$ we obtain that u, v are a basis of W.

   (d) If $x \in W$, we have $x = au + bv$ if and only if $T(x) = aT(u) + bT(v)$. Hence, if $T(x) = (1, 1)$, we have $x \in W$ if and only if $a + b = 3a - 5b = 1$ which holds if and only if $a = 3/4, b = 1/4$. Thus $x_n = 3^{n+1}/4 + (-5)^n/4$.

5. (a) The zero function is in W since $0' = 0$ and so $0'' + 2 \cdot 0' - 15 \cdot 0 = 0$. If $f, g \in W$ and $a, b \in \mathbb{R}$ we have $(af + bg)'' + 2(af + bg)' - 15(af + bg) = af'' + bg'' + 2af' + 2bg' - 15af - 15bg = a(f'' + 2f' - 15f) + b(g'' + 2g' - 15g) = a \cdot 0 + b \cdot 0 = 0$ which shows that $af + bg \in W$ and hence that W is a subspace of V.

   (b) If $u(x) = e^{3x}$, $v(x) = e^{-5x}$ we have $u' = 3u, u'' = 9u$ and so $u'' + 2u' - 15u = 0$. Similarly, $v' = -5v, v'' = 25v$ and so $v'' + 2v' - 15v = 0$. If $au + bv = 0$ then $au' + bv' = 0$ and $3au - 5bv = 0$. Evaluating at $x = 0$ we get $a + b = 3a - 5b = 0$ from which $a = b = 0$ showing that u, v are linearly independent.

   (c) We have $T(af + bg) = (af(0) + bg(0), af'(0) + bg'(0)) = aT(f) + bT(g)$. Hence T is linear. Since $\dim(W) = 2$ the functions u, v above are a basis of W and so every $f \in W$ can be uniquely written in the form $f = au + bv$ with a, b scalars. Now $T(f) = (au(0) + bv(0), au'(0) + bv'(0)) = (a + b, 3a - 5b)$ which shows that T is onto and one-to-one since the system of equations $a + b = x, 3a - 5b = y$ has a unique solution for a, b for any choice of $x, y \in \mathbb{R}$.

   (d) We have $a + b = 1, 3a - 5b = 1$ if and only if $a = 3/4, b = 1/4$. Thus, if we set $f = 3u/4 + v/4$, i.e., $f(x) = 3e^{3x}/4 + e^{-5x}/4$, we have $f'' + 2f' - 15f = 0$ and $f(0) = f'(0) = 1$.