McGill University
Math 223B: Linear Algebra
Solution Sheet for Assignment 2

1. a) $\{f \in V \mid f(2) = f(3) - 4\}$ is not a subspace since it doesn't contain the zero-function.

   b) $U = \{f \in V \mid f(2) = f(3) - f(4)\}$ is a subspace. Proof by subspace test: $\mathbf{0} \in U$ clear.
   Let $f, g \in U$; then $(f+g)(2) = f(2) + g(2) = f(3) - f(4) + g(3) - g(4) = f(3) - f(4) + (g(3) - g(4)) = (f+g)(3) - (f+g)(4)$, so $f + g \in U$.
   Let $r \in \mathbb{R}$ and $f \in U$; then $(rf)(2) = r(f(2)) = r(f(3) - f(4)) = r(f(3)) - r(g(4)) = (rf)(3) - (rf)(4)$, so $rf \in U$.

   c) $U = \{f \in V \mid f(2) \leq f(3)\}$ is not a subspace because $U$ contains the function $f(x) = x$ but the function $(-1)f = -f = -x$ is not contained in $U$.

   d) $U = \{f \in V \mid f'(2) = 0\}$ is a subspace. Subspace test: $\mathbf{0} \in U$ clear.
   Suppose $f, g \in U$, then $(f+g)'(2) = (f'+g')(2) = f'(2) + g'(2) = 0 + 0 = 0$, so $f + g \in U$.
   Let $r \in \mathbb{R}$ and $f \in U$; then $(rf)'(2) = (rf')(2) = r(f'(2)) = r \cdot 0 = 0$, so $rf \in U$.

2.
$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = s_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + s_2 \begin{pmatrix} 0 \\ -5 \\ 5 \\ -8 \end{pmatrix} + s_3 \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix} + s_4 \begin{pmatrix} 7 \\ 0 \\ -2 \\ 4 \end{pmatrix}$$

leads to the system of linear equations

$$\begin{array}{ccccl} s_1 & & +3s_3 & +7s_4 & = 0 \\ & -5s_2 & +3s_3 & & = 0 \\ & 5s_2 & +3s_3 & -2s_4 & = 0 \\ s_1 & -8s_2 & +3s_3 & +4s_4 & = 0 \end{array}$$

Solving this system by Gauss elimination shows $s_1 = s_2 = s_3 = s_4 = 0$, so the set is linearly independent.

3. $(4, 20, 14) = s_1(2, 3, 4) + s_2(8, -9, 7) + s_3(0, 7, 3)$ leads to the system of linear equations

$$\begin{array}{cccl} 2s_1 & +8s_2 & & = 4 \\ 3s_3 & -9s_2 & +7s_3 & = 20 \\ 4s_1 & +7s_2 & +3s_3 & = 14 \end{array}$$

By Gaussian elimination this can be transformed into the system

$$\begin{array}{cccl} s_1 & & +\frac{4}{3}s_3 & = \frac{14}{3} \\ & s_2 & -\frac{1}{3}s_3 & = -\frac{2}{3} \end{array}$$

Choosing for example $s_3 = 2$ leads to $s_2 = 0$ and $s_1 = 2$, which gives the representation $(4, 20, 14) = 2(2, 3, 4) + 2(0, 7, 3)$. Choosing $s_3 = 0$ leads $s_2 = -\frac{2}{3}$ and $s_1 = \frac{14}{3}$, which gives the representation $(4, 20, 14) = \frac{14}{3}(2, 3, 4) - \frac{2}{3}(8, -9, 7)$.

$(4, 8, 16) = s_1(2, 3, 4) + s_2(8, -9, 7) + s_3(0, 7, 3)$ leads to the system of linear equations

$$\begin{array}{cccl} 2s_1 & +8s_2 & & = 4 \\ 3s_3 & -9s_2 & +7s_3 & = 8 \\ 4s_1 & +7s_2 & +3s_3 & = 16 \end{array}$$

which has no solution (Gauss elimination). Thus $(4,8,16)$ doesn't lie in the span of $M$.

$M$ cannot be linearly independent because $(4,20,14)$ has two different representations as a linear combination.

4. Let $\mathbf{0} = s_1 5x^4 + s_2 \sin(3x) + s_3 e^{-x}$. Plugging in $x=0$ we obtain $0 = s_1 \cdot 0 + s_2 \cdot 0 + s_3 \cdot 1 = s_3$. So $s_3 = 0$ and therefore $\mathbf{0} = s_1 5x^4 + s_2 \sin(3x)$. Plugging in $x=\pi$ we obtain $0 = s_1 5\pi^4 + s_2 \cdot 0$. Thus $s_1 = 0$ and therefore $\mathbf{0} = s_2 \sin(3x)$. Plugging in $x = \frac{\pi}{2}$ we obtain $0 = -s_2$, whence $s_2 = 0$.

   So $\{5x^4,\ \sin(3x),\ e^{-x}\}$ is linearly independent.

5. a) Let $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. Then $p'(x) = 3a_3x^2 + 2a_2x + a_1$ and $p'(0) = a_1$. So $W = \{a_3x^3 + a_2x^2 + a_0 \ :\ a_3, a_2, a_0 \in \mathbb{R}\}$.
   Every linear combination of $x^3$, $x^2$, $x^3 - 5x^2$ and 4, is a polynomial of degree at most 3 without $x$-term, thus $\operatorname{span}(M) \subseteq W$. Conversely every $a_3x^3 + a_2x^2 + a_0 \in W$ can be written as $a_3x^3 + a_2x^2 + \frac{a_0}{4}4$, so it's a linear combination of $x^3$, $x^2$ and 4. Thus $W \subseteq \operatorname{span}\{x^3, x^2, 4\} \subseteq \operatorname{span}(M)$.
   $M$ cannot be a basis since $x^3 - 5x^2$ is a linear combination of $x^3$ and $x^2$. But $\{x^3,\ x^2,\ 4\}$ is linearly independent, so $\dim(W) = 3$.

   b)

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \text{ satisfies } A^\top = -A \text{ if and only if } A = \begin{pmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{pmatrix}.$$

$$\text{Then } A = b\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

$$\text{So } \mathcal{B} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\} \text{ spans } W.$$

   Furthermore $A = \mathbf{0}$ in the linear combination above if and only if $b = c = f = 0$. Hence $\mathcal{B}$ is linearly independent. So $\mathcal{B}$ is a basis of $V$ and $\dim(V) = 3$.