McGill University
Math 223B: Linear Algebra
Solution Sheet for Assignment 1

1. Row reducing the augmented matrix of the system to reduced echelon form, we get

$$\begin{bmatrix} 2 & -3 & 6 & 2 & -5 & 2 \\ 2 & -2 & 2 & 3 & -5 & 0 \\ -2 & 2 & -2 & -4 & 8 & 2 \\ 5 & -6 & 9 & 7 & -14 & 1 \end{bmatrix} \begin{array}{c} \longrightarrow \\ R_2 = R_2 - R_1 \\ R_3 = R_3 + R_1 \\ R_4 = R_4 - 2R_1 \end{array} \begin{bmatrix} 2 & -3 & 6 & 2 & -5 & 2 \\ 0 & 1 & -4 & 1 & 0 & -2 \\ 0 & -1 & 4 & -2 & 3 & 4 \\ 1 & 0 & -3 & 3 & -4 & -3 \end{bmatrix} R_1 \leftrightarrow R_4 \begin{bmatrix} 1 & 0 & -3 & 3 & -4 & -3 \\ 0 & 1 & -4 & 1 & 0 & -2 \\ 0 & -1 & 4 & -2 & 3 & 4 \\ 2 & -3 & 6 & 2 & -5 & 2 \end{bmatrix}$$

$$\begin{array}{c} \longrightarrow \\ \\ \\ R_4 = R_4 - 2R_1 \end{array} \begin{bmatrix} 1 & 0 & -3 & 3 & -4 & -3 \\ 0 & 1 & -4 & 1 & 0 & -2 \\ 0 & -1 & 4 & -2 & 3 & 4 \\ 0 & -3 & 12 & -4 & 3 & 8 \end{bmatrix} \begin{array}{c} \\ \longrightarrow \\ R_3 = R_3 + R_2 \\ R_4 = R_4 + 3R_2 \end{array} \begin{bmatrix} 1 & 0 & -3 & 3 & -4 & -3 \\ 0 & 1 & -4 & 1 & 0 & -2 \\ 0 & 0 & 0 & -1 & 3 & 2 \\ 0 & 0 & 0 & -1 & 3 & 2 \end{bmatrix} \begin{array}{c} R_1 = R_1 + 3R_3 \\ R_2 = R_2 + R_3 \\ \longrightarrow \\ R_4 = R_4 - R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 & 3 \\ 0 & 1 & -4 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \\ \longrightarrow \\ R_3 = -R_3 \\ \\ \end{array} \begin{bmatrix} 1 & 0 & -3 & 0 & 5 & 3 \\ 0 & 1 & -4 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the given system has the same solution set as the system

$$\begin{aligned} x_1 - 3x_3 + 5x_5 &= 3 \\ x_2 - 4x_3 + 3x_5 &= 0 \\ x_4 - 3x_5 &= -2. \end{aligned}$$

Setting $x_3 = s, x_5 = t$ and solving for x_1, x_2, x_4 we get as a parametric description for the solution set

$$x_1 = 3 + 3s - 5t,\ x_2 = 4s - 3t,\ x_3 = s,\ x_4 = -2 + 3x_5,\ x_5 = t.$$

2. Since $A^2 = 100I$ we have $A(\frac{1}{100}A) = (\frac{1}{100}A)A = I$ which means that A is invertible with $A^{-1} = \frac{1}{100}A$. Writing our system in matrix form, we have

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \quad \text{which implies} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \frac{1}{100} A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{1}{2} \\ \frac{2}{5} \\ \frac{3}{10} \\ \frac{1}{10} \end{bmatrix}$$

which shows that the given system has the unique solution $x{=}1/5, x_2 = 1/2, x_3 = 2/5, x_4 = 3/10, x_5 = 1/10$.

3. We first have to bring A to upper triangular form using only row operations in which a multiple of a row is subtracted from a suceeding row.

$$A = \begin{bmatrix} 2 & 6 & -2 & 0 & 2 \\ 3 & 9 & -3 & 3 & 1 \\ -1 & -3 & 1 & -3 & 1 \end{bmatrix} \begin{array}{c} \longrightarrow \\ R_2 = R_2 - \frac{3}{2}R_1 \\ R_3 = R_3 + \frac{1}{2}R_1 \end{array} \begin{bmatrix} 2 & 6 & -2 & 0 & 2 \\ 0 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & -3 & 2 \end{bmatrix} \begin{array}{c} \\ \longrightarrow \\ R_3 = R_3 + R_2 \end{array} \begin{bmatrix} 2 & 6 & -2 & 0 & 2 \\ 0 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

If $L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$, $L_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, we have $L_3L_2L_1A = U$ and so $A = LU$ with

$$L = (L_3L_2L_1)^{-1} = L_1^{-1}L_2^{-1}L_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & -1 & 1 \end{bmatrix}.$$

To find a PLU decomposition for B we first bring B to upper triangular form using only interchanges of rows and adding multiples of rows to suceeding rows. We have

$$B = \begin{bmatrix} 0 & -1 & 2 & 1 \\ -1 & 1 & 3 & 1 \\ 1 & -1 & -3 & 6 \\ 2 & -2 & -4 & 1 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} -1 & 1 & 3 & 1 \\ 0 & -1 & 2 & 1 \\ 1 & -1 & -3 & 6 \\ 2 & -2 & -4 & 1 \end{bmatrix} \begin{array}{c} \longrightarrow \\ \\ R_3 = R_3 + R_1 \\ R_4 = R_4 + 2R_1 \end{array} \begin{bmatrix} -1 & 1 & 3 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 2 & 3 \end{bmatrix} R_3 \leftrightarrow R_4 \begin{bmatrix} -1 & 1 & 3 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 7 \end{bmatrix} = U$$

and so $P_2L_2L_1P_1B = U$ with

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Since P_2 is obtained form the identity matrix by permuting rows after the first row, namely the third and fourth rows, we have

$$P_2L_2 = L_2'P_2 \quad \text{with} \quad L_2' = P_2L_2P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad P_2L_1 = L_1'P_2 \quad \text{with} \quad L_1' = P_2L_1P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

and so $L_2'L_1'P_2P_1B = U$. Thus $B = PLU$ with

$$P = P_1^{-1}P_2^{-1} = P_1P_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad L = (L_1')^{-1}(L_2')^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

Since $\det(P) = \det(P_1)\det(P_2) = (-1)^2 = 1$, $\det(L) = 1$ and $\det(U) = 14$, we have $\det(B) = \det(P)\det(L)\det(U) = 14$.

4. (i) We first solve the system by gaussian elimination.

$$\begin{bmatrix} 1+i & 3 & 0 & i \\ 1 & 1-2i & -i & 0 \\ 2 & -4i & 1-i & 1 \end{bmatrix} \begin{matrix} \\ R_1 \leftrightarrow R_2 \\ \\ \end{matrix} \begin{bmatrix} 1 & 1-2i & -i & 0 \\ 1+i & 3 & 0 & i \\ 2 & -4i & 1-i & 1 \end{bmatrix} \begin{matrix} \longrightarrow \\ R_2 = R_2 - (1+i)R_1 \\ R_3 = R_3 - 2R_1 \end{matrix} \begin{bmatrix} 1 & 1-2i & -i & 0 \\ 0 & i & -1+i & i \\ 0 & -2 & 1+i & 1 \end{bmatrix} \begin{matrix} \\ R_3 = -iR_3 \\ \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 1-2i & -i & 0 \\ 0 & 1 & 1+i & 1 \\ 0 & -2 & 1+i & 1 \end{bmatrix} \begin{matrix} \\ \longrightarrow \\ R_3 = R_3 + 2R_2 \end{matrix} \begin{bmatrix} 1 & 1-2i & -i & 0 \\ 0 & 1 & 1+i & 1 \\ 0 & 0 & 3+3i & 3 \end{bmatrix} \begin{matrix} \\ \longrightarrow \\ R_3 = (3+3i)^{-1}R_3 \end{matrix} \begin{bmatrix} 1 & 1-2i & -i & 0 \\ 0 & 1 & 1+i & 1 \\ 0 & 0 & 1 & \frac{1-i}{2} \end{bmatrix}.$$

The given system is thus equivalent to the system $x + (1-2i)y - iz = 0, y + (1+i)z = 1, z = \frac{1-i}{2}$. By backsubstitution we get $z = \frac{1-i}{2}, y = 1 - (1+i)z = 0, x = iz = \frac{1+i}{2}$.

(ii) Solving by Cramer's Rule, we have $x = \Delta_1/\Delta, y = \Delta_2/\Delta, z = \Delta_3/\Delta$ where

$$\Delta_1 = \begin{vmatrix} i & 3 & 0 \\ 0 & 1-2i & -i \\ 1 & -4i & 1-i \end{vmatrix} = 3, \quad \Delta_2 = \begin{vmatrix} 1+i & i & 0 \\ 1 & 0 & -i \\ 2 & 1 & 1-i \end{vmatrix} = 0$$

$$\Delta_3 = \begin{vmatrix} 1+i & 3 & i \\ 1 & 1-2i & 0 \\ 2 & -4i & 1 \end{vmatrix} = -3i, \quad \Delta = \begin{vmatrix} 1+i & 3 & 0 \\ 1 & 1-2i & -i \\ 2 & -4i & 1-i \end{vmatrix} = 3-3i.$$

Hence $x = \frac{1+i}{2}$, $y = 0$, $z = \frac{1-i}{2}$.

5. Expanding the determinant of A_n for $n > 2$ along the first column, we get

$$d_n = \begin{vmatrix} 3 & 2 & 0 & 0 & \dots & 0 & 0 \\ 1 & 3 & 2 & 0 & \dots & 0 & 0 \\ 0 & 1 & 3 & 2 & \dots & 0 & 0 \\ 0 & 0 & 1 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 3 & 2 \\ 0 & 0 & 0 & 0 & \dots & 1 & 3 \end{vmatrix} = 3 \begin{vmatrix} 3 & 2 & 0 & \dots & 0 & 0 \\ 1 & 3 & 2 & \dots & 0 & 0 \\ 0 & 1 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 3 & 2 \\ 0 & 0 & 0 & \dots & 1 & 3 \end{vmatrix} - \begin{vmatrix} 2 & 0 & 0 & \dots & 0 & 0 \\ 1 & 3 & 2 & \dots & 0 & 0 \\ 0 & 1 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 3 & 2 \\ 0 & 0 & 0 & \dots & 1 & 3 \end{vmatrix} = 3d_{n-1} - 2d_{n-2}$$

after expanding the last determinant along the first row. We have $d_1 = 3 = 2^{1+1} - 1$, $d_2 = 7 = 2^{2+1} - 1$. Assume that $d_k = 2^{k+1} - 1$ for $k < n$. Then $d_n = 3d_{n-1} - 2d_{n-2} = 3(2^n - 1) - 2(2^{n-1} - 1) = 2^{n+1} - 1$. Hence, by the principle of finite induction, $d_n = 2^{n+1} - 1$ for all $n \geq 1$.