McGill University
Math 223 B: Linear Algebra
Assignment 5: due Wednesday March 31, 1999

1. If $A = \begin{bmatrix} 4 & 2 \\ -1 & 2 \end{bmatrix}$, find an invertible complex matrix P such that $P^{-1}AP$ is a diagonal matrix.

2. Let A be the matrix
$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

   (a) Find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix.

   (b) Find symmetric matrices Q, R such that $Q^2 = Q$, $R^2 = R$, $QR = RQ = 0$ and $A = Q+5R$. Use this to compute A^n.

   (c) Find a symmetric matrix B with $B^2 = A$.

3. Let V be the vector space of real 2×2 matrices and let T be the mapping of V into itself defined by $T(X) = AX - XA$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

   (a) Show that T is linear;

   (b) Find bases for the kernel and image of T.

   (c) Find the rank and nullity of T. Is T an isomorphism?

4. Let T be the linear mapping in problem 3.

   (a) Find the matrix of T with respect to the basis
$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

   (b) Find the matrix of T with respect to the basis
$$F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ F_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ F_3 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},\ F_4 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

   (c) If A and B are the matrices found in (a) and (b) respectively, find a matrix P such that $P^{-1}AP = B$.

5. Find a 4×4 matrix whose column space and null space are spanned by the matrices
$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix},\ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

   Hint: Using a suitable basis of $\mathbb{R}^4$, first find a linear operator on $\mathbb{R}^4$ having kernel and image spanned by the above vectors.