McGill University
Math 223B: Linear Algebra
Assignment 3: due Wednesday, February 17, 1999

1. Find bases for the rowspace, column space and null space of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 1 & -3 & 2 & 8 & 11 & -2 \\ 3 & -8 & 4 & 12 & 17 & -1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}.$$

   Find the rank and nullity of A.

2. Show that the vectors

$$(3, 7, 11, 15, 7, -5),\ (1, 3, 5, 7, 4, -3),\ (2, 2, 2, 2, -3, 4)$$

   form a basis for the solution space of the linear homogeneous system:

$$\begin{aligned} 3x_1 - 5x_2 + x_3 + x_4 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ 5x_1 - 5x_2 + x_3 + 2x_5 + x_6 &= 0 \\ 6x_1 - 6x_2 + x_4 + 2x_5 + x_6 &= 0 \\ 2x_1 - x_4 + 2x_5 + x_6 &= 0. \end{aligned}$$

3. Let V be the vector space of real 2×2 matrices and let W be the subspace spanned by the matrices

$$\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix},\ \begin{bmatrix} 2 & -5 \\ -6 & 6 \end{bmatrix},\ \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix},\ \begin{bmatrix} 1 & -6 \\ -5 & 1 \end{bmatrix}.$$

   (a) Find a basis for W and complete this basis to a basis of V.

   (b) Find a basis for the subspace of W consisting of those matrices in W which have trace zero.

4. Let V be the vector space of infinite sequences $x = (x_0, x_1, \ldots, x_n, \ldots)$ of real numbers and let $W = \{x \in V \mid x_{n+2} = 15x_n - 2x_{n+1} \text{ for } n \geq 0\}$.

   (a) Show that W is a subspace of V;

   (b) Let $T : W \to \mathbb{R}^2$ be the mapping defined by $T(x) = (x_0, x_1)$. Show that T is an isomorphism of vector spaces and use this to compute the dimension of W.

   (c) Show that the two sequences $u = (1, 3, 9, \ldots, 3^n, \ldots)$, $v = (1, -5, 25, \ldots, (-5)^n, \ldots)$ form a basis of W.

   (d) If x is the element of W with $x_0 = x_1 = 1$, use the above to find a formula for x_n.

5. Let V be the vector space of infinitely differentiable real valued functions on $\mathbb{R}$ and let $W = \{f \in V \mid f'' + 2f' - 15f = 0\}$.

   (a) Show that W is a subspace of V.

   (b) Show that the functions e^{3x}, e^{-5x} are in W and that they are linearly independent.

   (c) Given that the dimension of W is 2, show that the mapping $T : W \to \mathbb{R}^2$ defined by $T(f) = (f(0), f'(0))$ is an isomorphism of vector spaces.

   (d) Use the above to find the function $f \in W$ satisfying $f(0) = f'(0) = 1$.