McGill University
Math 223B: Linear Algebra

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Attempt all Questions.
Explain clearly and justify all your work.
All questions are of equal value.
No Calculators Allowed.

1. (a) Find bases for the row space, column space and null space of the matrix

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 6 & 5 & 4 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 1 & -1 \end{bmatrix}.$$

   (b) Find a 4×4 matrix which has the same nullspace as the matrix A above but whose column space is spanned by $(1,0,0,1), (0,1,1,0)$.

2. Which of the following subsets W of the given vector space V are subspaces of V?

   (a) $V =$ real vector space of differentiable real valued functions on the real line,

$$W = \{f \in V \mid f'(x) = e^x f(x) \text{ for all } x \in \mathbb{R}\};$$

   (b) $V = \mathbb{R}^3, W = \{(x, y, z) \in V \mid z = xy\}$;

   (c) $V =$ complex vector space of infinite sequences of complex numbers,

$$W = \{(z_1, z_2, \dots, z_n, \dots) \in V \mid z_{n+2} = iz_{n+1} + nz_n \text{ for } n \geq 1\}.$$

3. Let V be the vector space of 2×2 matrices over $\mathbb{R}$ and let $T : V \to V$ be the mapping defined by

$$T(X) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} X - X \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

   (a) Show that T is linear;

   (b) Find the matrix of T with respect to the basis

$$E = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}.$$

   (c) Find bases for the kernel and image of T.

   (d) Show that $T^3 = 4T$ and use this to find the eigenvalues of T.

   (e) Find a basis for V such that the matrix of T with respect to this basis is a diagonal matrix.

4. Which of the following sets of functions are linearly independent over $\mathbb{R}$?

   (a) $e^x, xe^x, e^{2x}, xe^{2x}$;

   (b) $1, \cos(2x), \sin^2(x)$;

   (c) $1+x, (1+x)^2, (1+x)^3$;

5. Find the infinite sequence $x = (x_0, x_1, \ldots, x_n, \ldots)$ which satisfies

$$x_{n+3} - 7x_{n+2} + 15x_{n+1} - 9x_n = 1 \text{ for } n \geq 0$$

   and $x_0 = x_1 = x_2 = 1$.

6. Show that $< X, Y >= \text{trace}(X^t Y)$ is an inner product on the vector space $M_{2\times 2}$ of real 2×2 matrices. With respect to this inner product

   (a) find an orthonormal basis for the subspace W of $M_{2\times 2}$ spanned by the matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix};$$

   (b) find the orthogonal projection of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ on W and find its coordinates with respect to the basis found in (a).

7. Let A be the matrix $\begin{bmatrix} 3 & 2 & -2 \\ 2 & 3 & 2 \\ -2 & 2 & 3 \end{bmatrix}$.

   (a) Show that $A^2 - 4A - 5I = 0$;

   (b) Find the eigenvalues of A and a basis of each eigenspace;

   (c) Find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix;

   (d) Find a symmetric matrix B such that $B^3 = A$.