1. (a) If $a_n = \frac{(-1)^n x^{2n}}{n4^n}$ then $|a_{n+1}/a_n| = \frac{|x|^{2n+2}}{(n+1)4^{n+1}}\frac{n4^n}{|x|^{2n}} = \frac{|x|^2}{4}\frac{n}{n+1}$ which converges to $x^2/4$ as $n \to \infty$. Hence, by the ratio test, the series converges absolutely for $|x|^2/4 < 1$ or $|x| < 2$ and diverges for $|x|^2/4 > 1$ or $|x| > 2$. Hence the radius of convergence is 2. At $x = \pm 2$ the series is $\sum_1^\infty \frac{(-1)^n}{n}$ which converges by the alternating series test.

(b) If $a_n = \frac{(-1)^n x^{2n}}{n4^n}$ then $|a_{n+1}/a_n| = \frac{|x|^{3n+3}}{64^{n+1}\sqrt{n+2}}\frac{64^n\sqrt{n+1}}{|x|^{3n}} = \frac{|x|^3}{64}\sqrt{\frac{n+1}{n+2}}$ which converges to $|x|^3/64$ as $n \to \infty$. The series converges absolutely for $|x| < 4$ and diverges for $|x| > 4$. The radius of convergence is therefore 4. At $x = 4$ the series is $\sum_0^\infty \frac{1}{\sqrt{n+1}} = \sum_1^\infty \frac{1}{\sqrt{n}}$ which is a divergent p-series with $p = 1/2$. At $x = -4$ the series is $\sum_1^\infty \frac{(-1)^n}{\sqrt{n}}$ which is convergent by the alternating series test.

2. (a) We have $\sin(x) = \sum_0^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ so that $\frac{\sin(t^2)}{t} = \sum_0^\infty (-1)^n \frac{x^{4n+1}}{(2n+1)!}$ so that

$$F(x) = \sum_0^\infty (-1)^n \frac{x^{4n+2}}{(4n+2)(2n+1)!} = \frac{x^2}{2} - \frac{x^6}{18} + \frac{x^{10}}{50} - \cdots.$$

(b) Since the series for $\sin(x)$ converges for all x and the operations performed do not change the radius of convergence, the series for $F(x)$ converges for all x.

(c) To 6 decimals we have $F(.02) = (.04)/2 - (.02)^6/18$ since $(.02)^{10}/50 < .00000011$.

3. (a) Let $r = (t^3/3, 2t, 2/t)$. Then $\frac{dr}{dt} = (t^2, 2, -2/t^2)$ and $|\frac{dr}{dt}| = (t^4+2)/t^2$ so that $\mathbf{T} = (\frac{t^4}{t^4+2}, \frac{2t^2}{t^4+2}, \frac{-2}{t^4+2})$. Hence $= \frac{4t}{(t^4+2)^2}(2t^2 2 - t^4, 2t^2)$, $\kappa = |\frac{d\mathbf{T}}{dt}|/|\frac{dr}{dt}| = \frac{4t^3}{(t^4+2)^2}$ and $\mathbf{N} = \frac{d\mathbf{T}}{dt}/|\frac{d\mathbf{T}}{dt}| = (\frac{2t^2}{t^4+2}, \frac{2-t^4}{t^4+2}, \frac{-2t^2}{t^4+2})$.

(b) Since $r(1) = (1/3, 2, 2), r'(1) = (1, 2, -2)$, the tangent line at $(1/3, 2, 2)$ is $x = t + 1/3, y = 2t + 2, z = 2 - 2t$. Between the planes $z = 1$ and $z = 2$ we have 1 so that the length of that part of the curve is $\int_1^2 (t^2 + 1/t^2)\, dt = 17/6$.

4. We have $\frac{\partial u}{\partial x} = \frac{x}{r}\frac{du}{dr}, \frac{\partial u}{\partial y} = \frac{y}{r}\frac{du}{dr}, \frac{\partial u}{\partial z} = \frac{z}{r}\frac{du}{dr}$, so that

(a) $(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial u}{\partial z})^2 = \frac{x^2+y^2+z^2}{r^r}(\frac{du}{dr})^2 = (\frac{du}{dr})^2$,

(b) $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}) = \frac{1}{r}\frac{du}{dr}(x\vec{i} + y\vec{j} + z\vec{k})$.

5. (a) We have $\frac{\partial z}{\partial x} = 3e^y - 3x^2$, $\frac{\partial z}{\partial y} = 3xe^y - 3e^{3y}$ so that, at $(0,0)$, we have $\frac{\partial z}{\partial x} = 3$, $\frac{\partial z}{\partial y} = -3$. Hence the equations of the tangent plane and normal line at $(0, 0, -1)$ are respectively $z = -1 + 3x - 3y$ and $x = -3t, y = 3t, z = -1 + t$.

(b) The point (x, y) is a critical point of the function $f(x, y)$ if and only if $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. Now $\frac{\partial f}{\partial x} = 3e^y - 3x^2$ and $\frac{\partial f}{\partial y} = 3xe^y - 3e^{3y}$ so that (x, y) is a critical point if and only if $e^y = x^2$ and $x = e^{2y}$. These two equations have the unique solution $x = 1, y = 0$. Now $A = \frac{\partial^2 f}{\partial x^2} = -6x$, $B = \frac{\partial^2 f}{\partial x \partial y} = 3e^y$, $C = 3e^y - 9e^{3y}$ so that at the critical point $(1, 0)$ we have $A < 0$, $AC - B^2 = (-6)(-6) - 9 = 27 > 0$ which shows that $f(1, 0) = 1$ is a local maximum. Since $f(-3, 0) = 17$ the function f does not have a maximum at $(1, 0)$.

6. (a) We have $\nabla T = (3x^2y + z^3, 3y^2z + x^3, 3z^2x + y^3)$ and $\mathbf{u} = \overrightarrow{PQ} = (-1, 2, 2)$ so that, at $(2, -1, 0)$, we have $D_{\mathbf{u}T} = \nabla T \cdot \mathbf{u}/|\mathbf{u}| = (-12, 8, -1) \cdot (-1, 2, 2)/3 = 26/3$.

   (b) Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be the position of the mosquito at time t. Then $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ is the velocity of the mosquito at time t. We have $|\mathbf{v}| =$ speed of mosquito $=5$ and the direction of $\mathbf{v}$ is, up to sign, the gradient of f at $(2, -1, 0)$, namely $(8, -6, 0)/10 = (4, -3, 0)/5$ so that, at $(2, -1, 0)$, we have $\mathbf{v} = \pm((4, -3, 0)$. At time t, the temperature of the mosquito is $T(\mathbf{r}(t))$. The rate of change of the temperature of the mosquito per unit time is therefore

$$\frac{d}{dt}T(\mathbf{r}(t) = \nabla T(\mathbf{r}(t)) \cdot \mathbf{v}$$

   which, at the time the mosquito is at $(2, -1, 0)$, is $(-12, 8, -1) \cdot \pm(4, -3, 0) = \mp 72$. Since the mosquito is flying in the direction of increasing temperature, the rate must be positive so that $\mathbf{v} = (-4, 3, 0)$ and the rate is 72. (Things are getting hot for the mosquito!)

7. $\int_0^2 \int_0^{x^2/2} \frac{x}{(1+x^2+y^2)^2}\, dydx = \iint_R \frac{x}{(1+x^2+y^2)^2}\, dydx$ where R is the region $0 \le x \le 2, 0 \le y \le x^2/2$. Hence the given integral is equal to $\int_0^2 \int_0^{\sqrt{2y}} \frac{x}{(1+x^2+y^2)^2}\, dxdy = \frac{1}{2} \int_0^2 (\frac{1}{(y+1)^2} - \frac{1}{y^2+5})\, dy$. The rest is Cal II and is left to the reader.

8. Volume $= \displaystyle\int_{-\pi/2}^{\pi/2} \int_0^{\cos\theta} (1 - r^2) r \, dr d\theta.$