Solutions to Written Assignment 3

1. The point (x,y) is a critical point of the function $f(x,y)$ if and only if $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. Now $\frac{\partial f}{\partial x} = 3e^y - 3x^2$ and $\frac{\partial f}{\partial y} = 3xe^y - 3e^{3y}$ so that (x,y) is a critical point if and only if $e^y = x^2$ and $x = e^{2y}$. These two equations have the unique solution $x = 1, y = 0$. Now $A = \frac{\partial^2 f}{\partial x^2} = -6x$, $B = \frac{\partial^2 f}{\partial x \partial y} = 3e^y$, $C = 3e^y - 9e^{3y}$ so that at the critical point $(1,0)$ we have $A < 0$, $AC - B^2 = (-6)(-6) - 9 = 27 > 0$ which shows that $f(1,0) = 1$ is a local maximum. Since $f(-3,0) = 17$ the function f does not have a maximum at $(1,0)$.

2. Since the function $f(x,y) = 2x + 3y$ is continuous it has a maximum on the ellipse $x^2 + xy + 2y^2 = 37$. This maximum is a critical point of $L = 2x+3y-\lambda(x^2+xy+2y^2-37)$. Since $\frac{\partial L}{\partial x} = 2-\lambda(2x+y)$ and $\frac{\partial L}{\partial y} = 3-\lambda(x+4y)$, the critical points of l satisfy $\lambda(2x+y) = 2$, $\lambda(x+4y) = 3$, $x^2 + xy + 2y^2 = 37$. Eliminating λ in the first two equations gives $y = 4x/5$. Substituting this in the third equation gives $x = 5\sqrt{37/77}, y = 4\sqrt{37/77}$ and $x = -5\sqrt{37/77}, y = -4\sqrt{37/77}$. The function f has the maximum $f(5\sqrt{37/77}, 4\sqrt{37/77}) = 22\sqrt{37/77}$ on the given curve. It takes its minimum value $-22\sqrt{37/77}$ at $(-5\sqrt{37/77}, -4\sqrt{37/77})$.

3. Since the first integral is improper at the lower limit, we have

$$\begin{aligned}
\int_0^2 \int_x^2 \frac{\ln y}{\sqrt{y}}\, dy\, dx &= \lim_{\epsilon \to 0} \int_\epsilon^2 \int_x^2 \frac{\ln y}{\sqrt{y}}\, dy\, dx = \lim_{\epsilon \to 0} \int_\epsilon^2 \int_\epsilon^y \frac{\ln y}{\sqrt{y}}\, dy\, dx = \lim_{\epsilon \to 0} \int_\epsilon^2 (y-\epsilon)\frac{\ln y}{\sqrt{y}}\, dy \\
&= \lim_{\epsilon \to 0} \int_\epsilon^2 \sqrt{y} \ln y\, dy - \lim_{\epsilon \to 0} \epsilon \int_\epsilon^2 \frac{\ln y}{\sqrt{y}}\, dy \\
&= \lim_{\epsilon \to 0} \left(\frac{2}{3} y^{3/2} \ln y \Big|_\epsilon^2 - \frac{2}{3} \int_\epsilon^2 \sqrt{y}\, dy - 2\epsilon\sqrt{y} \ln y \Big|_\epsilon^2 + 2\epsilon \int_\epsilon^2 \frac{dy}{\sqrt{y}} \right) \\
&= \lim_{\epsilon \to 0} \left(\frac{2}{3} y^{3/2} \ln y - \frac{4}{9} y^{3/2} - 2\epsilon\sqrt{y} \ln y + 4\epsilon\sqrt{y} \right) \Big|_\epsilon^2 \\
&= \frac{4}{9}\sqrt{2}(3\ln 2 - 2), \text{ using l'Hospital's Rule } (\lim_{\epsilon \to 0} y^\alpha \ln y = 0 \text{ for } \alpha > 0).
\end{aligned}$$

4. Volume $= \displaystyle\int_0^{2\pi} \int_1^2 (r\cos\theta + 2) r\, dr\, d\theta = 6\pi.$