Solutions to Written Assignment 2

1. (a) Let \( r = (t^3/3, 2t, 2t) \). Then \( \frac{dr}{dt} = (t^2, 2, -2t^2) \) and \( |\frac{dr}{dt}| = (t^4 + 2)/t^2 \) so that \( T = (\frac{t^4}{t^4+2}, \frac{2t^2}{t^4+2}, \frac{-2t^2}{t^4+2}) \).

Hence = \( \frac{4t}{(t^4+2)^2}(2t^4 - t^4, 2t^2) \), \( \kappa = |\frac{dT}{dt}|/|\frac{dr}{dt}| = \frac{4t^2}{(t^4+2)^2} \) and \( N = \frac{dT}{dt}/|\frac{dr}{dt}| = (\frac{2t^2}{t^4+2}, \frac{-2t^2}{t^4+2}) \).

(b) Since \( r(1) = (1/3, 2, 2) \), \( r'(1) = (1, 2, -2) \), the tangent line at \((1/3, 2, 2)\) is \( x = t+1/3, y = 2t+2, z = 2-2t \).

2. We have \( \frac{\partial u}{\partial x} = \frac{e}{r} \frac{du}{dr}, \frac{\partial u}{\partial y} = \frac{y}{r} \frac{du}{dr}, \frac{\partial u}{\partial z} = -\frac{z}{r} \frac{du}{dr} \), so that

(a) \( (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial u}{\partial z})^2 = \frac{x^2+y^2+z^2}{r^2}(\frac{du}{dr})^2 = (\frac{du}{dr})^2 \),

(b) \( \nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}) = \frac{1}{r} \frac{du}{dr} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \).

3. (a) We have \( \frac{\partial x}{\partial z} = 3e^y - 3x^2, \frac{\partial y}{\partial z} = 3xe^y - 3e^y \) so that, at \((0, 0)\), we have \( \frac{\partial x}{\partial z} = 3, \frac{\partial y}{\partial z} = -3 \). Hence the equations of the tangent plane and normal line at \((0, 0, -1)\) are respectively \( z = -1 + 3x - 3y \) and \( x = -3t, y = 3t, z = -1 + t \).

(b) If \( r = (2r^2 + t^3, s^2 - t^3) \) we have \( \frac{\partial r}{\partial s} = (4s, 2st^3, 2s - t^3), \frac{\partial r}{\partial t} = (3t^2, 3s^2t^2, -3st^2) \) so that, when \( s = t = 1 \), we have \( r = (3, 1, 0), \frac{\partial r}{\partial s} = (4, 2, 1), \frac{\partial r}{\partial t} = (3, 3, -3) \). A normal vector to the tangent plane at \((3, 1, 0)\) is \( \frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} = (-9, 15, 6) \) so that the tangent plane at \((3, 1, 0)\) is \( 3x - 5y - 2z = 4 \).

4. (a) We have \( \nabla T = (3x^2y + z^3, 3y^2z + x^3, 3z^2x + y^3) \) and \( \mathbf{u} = \overrightarrow{PQ} = (-1, 2, 2) \) so that, at \((2, -1, 0)\), we have \( D_{uT} = \nabla T \cdot \mathbf{u} = |\mathbf{u}| = (-12, 8, -1) \cdot (-1, 2, 2)/3 = 26/3 \).

(b) Let \( r(t) = (x(t), y(t), z(t)) \) be the position of the mosquito at time \( t \). Then \( \mathbf{v} = \frac{dr}{dt} \) is the velocity of the mosquito at time \( t \). We have \( |\mathbf{v}| = \text{speed of mosquito} = 5 \) and the direction of \( \mathbf{v} \) is, up to sign, the gradient of \( f \) at \((2, -1, 0)\), namely \((8, -6, 0)/10 = (4, -3, 0)/5 \) so that, at \((2, -1, 0)\), we have \( \mathbf{v} = \pm(4, -3, 0) \). At time \( t \), the temperature of the mosquito is \( T(r(t)) \). The rate of change of the temperature of the mosquito per unit time is therefore

\[ \frac{d}{dt} T(r(t)) = \nabla T(r(t)) \cdot \mathbf{v} \]

which, at the time the mosquito is at \((2, -1, 0)\), is \((-12, 8, -1) \cdot \pm(4, -3, 0) = \mp 72 \). Since the mosquito is flying in the direction of increasing temperature, the rate must be positive so that \( \mathbf{v} = (-4, 3, 0) \) and the rate is 72. (Things are getting hot for the mosquito!)